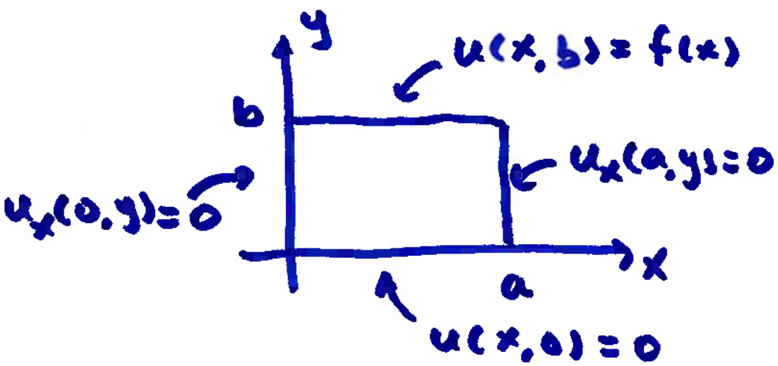
9.7 Laplace's Eq. (part 2)

$u_{xx} + u_{yy} = 0 \qquad 0 < x < a \qquad 0 < y < b$

there is a variety of BC's



$$\left.\begin{aligned} u(x,0) &= 0 \\ u_x(0,y) &= 0 \\ u_x(a,y) &= 0 \end{aligned}\right\}$$ homogeneous, used to solve X, Y

$u(x,b) = f(x)$ nonhomogeneous, used at the end for coefficients of series

$u(x,y) = X(x)Y(y) \qquad u_{xx} = X''Y \qquad u_{yy} = XY''$

$X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = C$$

C depends on BC's

$u(x,0) = 0 \rightarrow X(x)Y(0) = 0 \rightarrow Y(0) = 0$

$u_x(0,y) = 0 \rightarrow X'(0)Y(y) = 0 \rightarrow X'(0) = 0$

$u_x(a,y) = 0 \rightarrow X'(a)Y(y) = 0 \rightarrow X'(a) = 0$

X has two BC's $\rightarrow$ solve for X first

$X'' - cX = 0 \qquad X'(0) = X'(a) = 0$

check $c = 0$ → nontrivial X that meets BC's

$X'' = 0 \rightarrow X = c_1 x + c_2$

$X' = c_1 \qquad X'(0) = X'(a) = 0 \longrightarrow c_1 = 0$

so, $\boxed{X = 1}$ if $c = 0 \rightarrow \boxed{\lambda = 0}$

the BC's $X'(0) = X'(a) = 0$ suggest sine and cosine

so $c < 0 \rightarrow$ let $c = -\lambda$

$X'' + \lambda X = 0 \qquad X'(0) = X'(a) = 0$

Seen this before: $\boxed{\lambda_n = \frac{n^2\pi^2}{a^2}} \qquad \boxed{X_n = \cos\left(\frac{n\pi x}{a}\right)}$

$n = 1, 2, 3, \ldots$

also good for $\boxed{n = 0, 1, 2, 3, \ldots}$

and that includes $c = 0$ case

now solve $-\frac{Y''}{Y} = C = -\lambda = -\frac{n^2\pi^2}{a^2}$ $\quad n = 0, 1, 2, 3, \ldots$

find Y when $n=0$

$Y'' = 0 \rightarrow Y = c_1 y + c_2$ $\quad$ BC: $Y(0) = 0$

$0 = c_2$ $\quad$ so, $\boxed{Y = y}$ when $\boxed{\lambda = n = 0}$

$n > 0$ $\quad \frac{Y''}{Y} = \frac{n^2\pi^2}{a^2}$

$Y'' - \frac{n^2\pi^2}{a^2} Y = 0$ $\quad Y(0) = 0$

$Y = c_1 \cosh\left(\frac{n\pi y}{a}\right) + c_2 \sinh\left(\frac{n\pi y}{a}\right)$

$0 = c_1$ $\quad \boxed{Y_n = \sinh\left(\frac{n\pi y}{a}\right)}$ $\quad n = 1, 2, 3, \ldots$

$u_n = X_n Y_n$

for $n = 0$, $u_0 = (1)(y) = y$

for $n = 1, 2, 3, \ldots$ $\quad u_n = \sinh\left(\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right)$

$$\boxed{u(x,y) = \underbrace{\tfrac{1}{2}C_0}\, y + \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right)}$$

choose $\frac{1}{2}C_0$ for coeff for $n=0$ because we will end up w/ a cosine series

last BC: $u(x,b) = f(x)$

$$f(x) = \underbrace{\tfrac{1}{2}[C_0 b]}_{\text{"}a_0\text{"}} + \sum_{n=1}^{\infty} \underbrace{\left[C_n \sinh\left(\frac{n\pi b}{a}\right)\right]}_{\text{"}a_n\text{"}} \cos\left(\frac{n\pi x}{a}\right)$$

cosine series w/ coeff "a_0" and "a_n"

cosine series coeff: $\frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi x}{a}\right)dx$

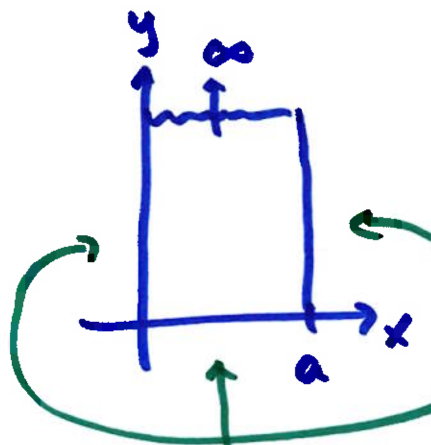
a

L for x is a

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx \qquad a_0 = \frac{2}{a}\int_0^a f(x)\,dx$$

$$C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx \qquad C_0 = \frac{2}{ab}\int_0^a f(x)\,dx$$

we can model a very long plate by using semi-infinite domain



$0 < x < a$

let y go to ∞ impose some kind of BC there

as an example, let's insulate left and right

$u_x(0,y) = 0$

$u_x(a,y) = 0$

$u(x,0) = f(x)$

u is bounded as $y \to \infty$ this is a BC at ∞

$u = XY$ X solution is identical to previous example

$\boxed{X_n = \cos\left(\frac{n\pi x}{a}\right)}$ $\boxed{\lambda_n = \frac{n^2\pi^2}{a^2}}$ $n = 0, 1, 2, 3, \ldots$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$Y'' - \lambda Y = 0 \to Y'' - \frac{n^2\pi^2}{a^2} Y = 0$$

$$Y = c_1 e^{n\pi y/a} + c_2 e^{-n\pi y/a}$$

exponentials are more convenient when dealing w/ ∞

BC: u bounded as $y \to \infty$ $\quad u = XY \to Y$ bounded as $y \to \infty$

so, $c_1 = 0$ $\quad \boxed{Y_n = e^{-n\pi y/a}}$ $\quad n = 0, 1, 2, 3, \ldots$

$u = XY$

$u_0 = 1$ $\quad$ (X_0 and Y_0 are both 1)

$u_n = e^{-n\pi y/a} \cos\left(\frac{n\pi x}{a}\right)$

$$\boxed{u(x,y) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n e^{-n\pi y/a} \cos\left(\frac{n\pi x}{a}\right)}$$

last BC: $u(x,0) = f(x)$

$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{a}\right)$ $\quad$ cosine series

$$\boxed{a_0 = \frac{2}{a} \int_0^a f(x)\, dx}$$

$$\boxed{a_n = \frac{2}{a} \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx}$$