7.1 Laplace Transform

type of integral transform (another common one is the Fourier transform)

Laplace Transform of $f(t)$ is

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st}\, dt}$$

kernel of integral transform

example $f(t) = 1$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}\, dt$$

treat s as constant for integration

$$= \lim_{a\to\infty} \int_0^a e^{-st}\, dt = \lim_{a\to\infty} \frac{1}{-s} e^{-st} \Big|_0^a \; t$$

$$= \lim_{a\to\infty} \left(-\frac{1}{s} \underbrace{e^{-sa}}_{\substack{\text{goes to } 0 \\ \text{if } s>0}} + \frac{1}{s} e^0 \right) = \boxed{\frac{1}{s}, \quad s>0}$$

<u>example</u> $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty t \cdot e^{-st}\,dt$$

$$= \lim_{a\to\infty} \int_0^a t \cdot e^{-st}\,dt$$

by parts

$u = t \qquad dv = e^{-st}\,dt$

$du = dt \qquad v = -\frac{1}{s}e^{-st}$

$$= \lim_{a\to\infty} \left(-\frac{t}{s} e^{-st} \Big|_{t=0}^{t=a} + \int_0^a \frac{1}{s} e^{-st}\,dt \right)$$

$$= \lim_{a\to\infty} \left(-\frac{t}{s} e^{-st} \Big|_{t=0}^{t=a} - \frac{1}{s^2} e^{-st} \Big|_{t=0}^{t=a} \right)$$

$$= \lim_{a\to\infty} \left(-\frac{a}{s} \underbrace{e^{-sa}}_{\substack{\to 0 \\ s>0}} - \frac{1}{s^2} \underbrace{e^{-sa}}_{\substack{\to 0 \\ s>0}} + \frac{1}{s^2} \right) = \boxed{\frac{1}{s^2}, \quad s>0}$$

so $\mathcal{L}\{t\} = \frac{1}{s^2}, \; s>0$

$\mathcal{L}\{c_1 + c_2 t\}$ $\quad$ c_1, c_2 constants

$$= \int_0^\infty (c_1 + c_2 t)\, e^{-st}\, dt$$

$$= \int_0^\infty c_1 \cdot e^{-st}\, dt + \int_0^\infty c_2 t \cdot e^{-st}\, dt$$

$$= c_1 \underbrace{\int_0^\infty 1 \cdot e^{-st}\, dt}_{\mathcal{L}\{1\}} + c_2 \underbrace{\int_0^\infty t \cdot e^{-st}\, dt}_{\mathcal{L}\{t\}}$$

so, $\mathcal{L}\{c_1 + c_2 t\} = c_1 \mathcal{L}\{1\} + c_2 \mathcal{L}\{t\}$ $\quad$ LT is <u>linear</u>

$$\mathcal{L}\{c_1 \cdot f(t) + c_2 \cdot g(t)\}$$

$$= c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}$$

$$= c_1 F(s) + c_2 G(s)$$

provided these exist

BUT $\mathcal{L}\{f(t) \cdot g(t)\}$

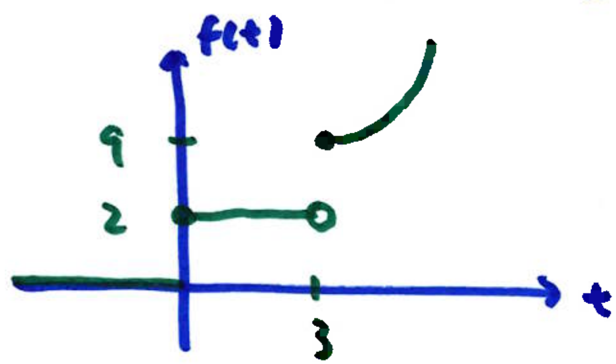
$\neq \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}$

in general

as long as $f(t)$ is $\underbrace{\text{piecewise continuous}}_{\text{finite number of discontinuities}}$, then $\mathcal{L}\{f(t)\}$ exists

example

$$f(t)=\begin{cases} 0 & t<0 \\ 2 & 0\le t<3 \\ t^2 & t\ge 3 \end{cases}$$



$F(s)=?$

$$F(s)=\int_0^\infty f(t)\,e^{-st}\,dt=\int_0^3 2\cdot e^{-st}\,dt+\int_3^\infty t^2\cdot e^{-st}\,dt$$

$$=\dots=\frac{2-2e^{-3s}}{s}+\frac{e^{-3s}(3s+1)}{s^2}\qquad (s>0)$$

in practice, we don't usually do LT by hand example $\mathcal{L}\{3t^4 - e^{5t} + 2\sin 3t\}$

$$= 3\underbrace{\mathcal{L}\{t^4\}}_{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}} - \mathcal{L}\{e^{5t}\} + 2\mathcal{L}\{\sin 3t\}$$

from table

$$= 3 \cdot \frac{4!}{s^5} - \frac{1}{s-5} + 2 \cdot \frac{3}{s^2+9} \quad , \quad s > 5$$

inverse transform: $\mathcal{L}^{-1}\{F(s)\} = f(t)$

use table from right to left

$$\mathcal{L}^{-1}\left\{\frac{2-4s}{s^2+16}\right\} = \mathcal{L}^{-1}\underbrace{\left\{\frac{2}{s^2+16}\right\}}_{\text{looks like}} - 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}$$

$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{4}{s^2+4^2}\right\} - 4\mathcal{L}^{-1}\{\cdots\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} - \cdots = \frac{1}{2}\sin kt + \cdots$$

$f(t)$	$F(s)$	
1	$\dfrac{1}{s}$	$(s > 0)$
t	$\dfrac{1}{s^2}$	$(s > 0)$
$t^n \quad (n \geqq 0)$	$\dfrac{n!}{s^{n+1}}$	$(s > 0)$
$t^a \quad (a > -1)$	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	$(s > 0)$
e^{at}	$\dfrac{1}{s-a}$	$(s > a)$
$\cos kt$	$\dfrac{s}{s^2+k^2}$	$(s > 0)$
$\sin kt$	$\dfrac{k}{s^2+k^2}$	$(s > 0)$
$\cosh kt$	$\dfrac{s}{s^2-k^2}$	$(s > \|k\|)$
$\sinh kt$	$\dfrac{k}{s^2-k^2}$	$(s > \|k\|)$
$u(t-a)$	$\dfrac{e^{-as}}{s}$	$(s > 0)$