Warning: incomplete

Vocab

- Dimension of Null space : # of free Variables
- Dimension of Rank : # pivot columns

$$\underbrace{R + N = \overset{\text{columns of given matrix}}{n}}_{\text{Does Rank + Null = span?}}$$

- Linearly independence
  - det ≠ 0
  - Lin. IDP columns : Original matrix columns of columns with pivots

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \underline{7} & \underline{8} & \underline{10} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{bmatrix} \rightarrow \begin{bmatrix} \textcircled{1} & 2 & 3 \\ 0 & \textcircled{-3} & -6 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

IDP columns

  - Lin. IDP Rows : Echelon rows that have pivots

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{bmatrix} \rightarrow \left. \begin{bmatrix} \textcircled{1} & 2 & 3 \\ 0 & \textcircled{-3} & -6 \\ 0 & 0 & \textcircled{1} \end{bmatrix} \right\} \text{Lin. IDP Rows}$$

  - A^T : Flip columns and rows

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$$

- Basis Vectors : originals matrix columns that align with the reduced matrixes pivots
  - ↳ Each vector requires a pivot

Asking for columns : use original

Asking for rows : use reduced

- Span : linear Collection of Vectors which are multiples of basis vectors

- Infinite Solutions : matrix has free Variables

- Sub-space :
  - Closed under addition and scalar multiplication

determinant rules:

$\det(A^{-1}) = \frac{1}{\det(A)}$

$\det(kA) = k^N \det(A)$ N: # rows in matrix

$\det(A^T) = \det(A)$

- Matrix operations:
  - Adding to row doesn't change det
  - Multiplying row or column by something, multiply det
  - Each row switch multiply det by (-1)
- $\det(A) = 0$ → means there is a zero row

Only square matrix:

- Invertable → inverse
- Can find determinant

Other equations:

- $\dim(\text{rank}) + \dim(\text{null}) = \dim(\text{col})$
- $\text{Rank}(A^T) = \text{Rank}(A)$

Lesson 15-30 (3.4-6.1)

3.4 Matrix Operations

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad a_{\underset{\uparrow \atop \text{Row}}{1}\overset{\text{column} \atop \downarrow}{2}}$$

· Scalar Multiplication:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow 3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

· Subtraction

$$A\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - B\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + B(-1)\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & 3 \end{bmatrix}$$

· Vectors in Matrix

Vector: 1 column or row of matrix

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Ex)

$$\left.\begin{aligned} 2x_1 + 3x_2 - 4x_3 &= 1 \\ -x_2 + 5x_3 &= 2 \\ 3x_1 + 4x_2 + 0 &= 3 \end{aligned}\right\} x_1\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + x_1\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} -4 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

· Matrices multiplication

Row x column

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 24 \\ 14 \end{bmatrix}$$

$\underline{2} \times 3 \leftarrow$ Must match to multiply $\rightarrow 3 \times \underline{1}$

Tells you new Matrix size

Multiply elements of row of focus to each column element

∴ · Start with 1st A row then work through columns of B matrix, down each column then to next column

· once all columns compleated move to next row of A and work through all B columns again

Ex)

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}} \times \overset{B}{\begin{bmatrix} -4 & 0 & 3 \\ 1 & -5 & 2 \end{bmatrix}} = \begin{bmatrix} \overset{R_{11}C_{11}}{-4(1)} + \overset{R_{12}C_{21}}{1(1)} & 1(0) + 1(-5) & 1(3) + 1(2) \\ \underset{R_{21}C_{11}}{2(-4)} + \underset{R_{22}C_{21}}{1(1)} & 2(0) + 1(-5) & 2(3) + 1(2) \end{bmatrix} = \begin{bmatrix} -3 & -5 & 5 \\ -7 & -5 & 8 \end{bmatrix}$$

· Matrix Equation: $A\vec{x} = \vec{b}$

Ex)

$$\begin{bmatrix} \underline{1} & 0 & -1 & 6 & 5 \\ 0 & \underline{1} & 9 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

↳ 2 pivots, 5 variables ∴ 3 free variables

\# Variables - # pivots = # free Variables

$x_5 = t \quad x_4 = s \quad x_3 = r$

$x_2 = -9x_3 + 2x_4 - 4x_5 = -9r + 2s - 4t$

$x_1 = r - 6s - 5t$

3.5 Inverse of Matrices

Identity matrix: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Everything zero, ones on main diagonal

$$\left.\text{If } A \text{ is } 2 \times 2 \text{ then } A^{-1} = \frac{1}{\underbrace{ad - bc}_{\text{Determinant } (\det(A))}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right\} \text{If } A^{-1} \text{ exists } \det(A) \neq 0$$

Inverse Acts as Matrix division:

Ex) $x_1 + 2x_2 = 5$
$3x_1 + 4x_2 = 6$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underset{2\times 2}{\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 5 \\ 6 \end{bmatrix}}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2(5)+1(6) \\ 3/2(5)+(-1/2)(6) \end{bmatrix} = \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$$

When $n \times n$ $n > 2$

$$\left[A \;\vdots\; I\right] \xrightarrow[\text{op.}]{\text{Row}} \left[I \;\vdots\; A^{-1}\right]$$

Ex)

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_1(-3)+R_2} \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{R_2(-\frac{1}{2})} \left[\begin{array}{cc:cc} 1 & 0 & \overbrace{-2}^{A^{-1}} & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

3.6-3.7 Determinants

Example: $A = \begin{bmatrix} 2 & 0 & 4 \\ 3 & 4 & 2 \\ 0 & 4 & -2 \end{bmatrix}$ Find det A

Pick any row or column to expand:

ex) Column 1

$(-1)^{i+j}$ i, j is row, col #

$$\det A = \underset{a_{11}}{(2)} \underbrace{\underbrace{\begin{vmatrix} 4 & 2 \\ 4 & -2 \end{vmatrix}}_{\text{Det. of matrix after blocking out row 1, column 1}}}_{\text{Cofactor of } a_{11}} \underbrace{- \underset{a_{21}}{(3)} \begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix}}_{\text{cofactor of } a_{21}} + (0)\begin{vmatrix} 0 & 4 \\ 4 & 2 \end{vmatrix} = (2)(-8-8)-(3)(0-16)+0(0-16) = \boxed{16}$$

Called: minor of a_{11}

Cofactor Sign $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ Signs switch every column and every row

Example

$$A = \begin{bmatrix} 8^+ & 0^- & 0^+ & 5^- \\ 5^- & 8^+ & -3^- & -7^+ \\ \boxed{2^+ \;\; 0^- \;\; 0^+ \;\; 0^-} \\ 7^- & 2^+ & 1^- & 7^+ \end{bmatrix}$$

$$\det A = (2)\underbrace{\begin{vmatrix} \boxed{0 \;\; 0 \;\; 5} \\ 8 & 3 & -7 \\ 2 & 1 & 7 \end{vmatrix}}_{\text{Expand again row 1 looks good}} - (0)\cancel{\begin{vmatrix} 8 & 0 & 5 \\ 5 & 3 & -7 \\ 7 & 1 & 7 \end{vmatrix}} \to 0 + \cancel{(0)\begin{vmatrix} \vdots \end{vmatrix}} \to 0 - \cancel{(0)\begin{vmatrix} \vdots \end{vmatrix}} \to 0$$

All others zero ∴ 5 matrix the only survivor

$$= (2)(5)\begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix}$$

↳ Full form: $(2)\left[0\begin{vmatrix} 3 & -7 \\ 1 & 7 \end{vmatrix} - 0\begin{vmatrix} 0 & 5 \\ 2 & 7 \end{vmatrix} + 5\begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix}\right]$

- Triangular matrix: $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ upper triangle

  $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ lower triangle

$\det\left(\begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{3} \end{bmatrix}\right) = (1)(3)-(0)(2) = \boxed{3}$

$A = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{4} & 5 \\ 0 & 0 & \boxed{6} \end{bmatrix}$ Along column 1:

$\det A = (1)\begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = (\boxed{1})(\boxed{4})(\boxed{6}) = 24$

Triangular or diagonal matrix: determinant is product of main diagonal elements

Gaussian elimination produces triangular matrix

1) use elimination,
2) produce triangular matrix
3) Find det

→ Rules:

1) Exchange two rows switch sign of determinant

Ex)

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det A = -2$

$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $\det B = 2$

2) Multiplying one row by K ($K \neq 0$) multiply determinant by same K

Ex)

$A = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ 3 & 4 \end{bmatrix} = \det A = \frac{2}{5} - \frac{3}{5} = -\frac{1}{5}$

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ 10 times row 1 of A

$\det B = -2 = 10(\det A)$

3) Multiply one row and adding to another doesn't change determinant

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det A = -2$

$\xrightarrow{(-3)R_1 + R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ Triangular, determinant is $(1)(-2) = -2$

Example

① $A = \begin{bmatrix} 2 & 4 & -2 & 6 \\ 1 & 2 & 5 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & -6 & 3 \end{bmatrix}$

② $\xrightarrow{\text{Swap } (R_1, R_3)} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 5 & 4 \\ 2 & 4 & -2 & 6 \\ 0 & 2 & -6 & 3 \end{bmatrix}$ determ. sign change

③ No affect on det A $\left\{ \xrightarrow[(-2)R_1 + R_3]{(-1)R_1 + R_2} \right.$ $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & -6 & -2 \\ 0 & 2 & -6 & 3 \end{bmatrix}$

④ $\xrightarrow[(-2)R_2 + R_4]{(-2)R_2 + R_3} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -12 & -2 \\ 0 & 0 & -12 & 3 \end{bmatrix}$

⑤ $\xrightarrow{(-1)R_3 + R_4} \underbrace{\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -12 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}}$

determinant is $(1)(1)(-12)(5) = -60$

one swap earlier so $\det A = -(-60) = \boxed{60}$

Transpose: Switch column and row vectors

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 15 \end{bmatrix}$

4.1 The Vector Space R^3

Linear Dependent: one vector is scalar multiple of the other

Linearly independent: If $a = b = 0$ then $a\vec{u} + b\vec{v} = 0$ is only solution

Example: $\vec{u} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}$

Are they linearly independent?

If not write one as a linear combination of the other two

Ways to solve:

1) unique solution only if $\begin{bmatrix} 4 & -5 & 0 \\ 0 & 1 & -4 \\ 1 & -1 & -1 \end{bmatrix}^{-1}$ exists

$\longrightarrow$ determinant $\neq 0$

2) Reduce augmented matrix $\begin{bmatrix} 4 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 4 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 4 & -5 & 0 & 0 \end{bmatrix} \xrightarrow{4R_1+R_3} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ \boxed{0 \quad 0 \quad 0 \quad 0} \end{bmatrix}$$

Infinitely many solutions
BUT linearly independent only
if $a=b=c=0$ is ONLY solution

So $\vec{u}, \vec{v}, \vec{w}$ are NOT linearly independent

From reduced matrix: $c=r$ (free, no pivot in column 3)

$b-4c=0 \longrightarrow b=4c=4r$

$a-b-c=0 \longrightarrow a=b+c=5r$

$a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$ Choose $r=1 \longrightarrow a=5, \ b=4, \ c=1$

$5\vec{u}+4\vec{v}+\vec{w}=0$

$\boxed{\vec{w}=-5\vec{u}-4\vec{v}}$ or $\boxed{\vec{u}=-\frac{4}{5}\vec{v}-\frac{1}{5}\vec{w}}$

4.2 The Vector Space R^n and Subspace

$R_1=[x] \rightarrow R_2=\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow R_3=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \ldots \rightarrow R_n$ contains all the subspaces that came before it

- Closed under addition: pick any two vectors in subspace - their sum MUST remain in subspace

Ex) $\underbrace{\vec{u}=\begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} \quad \vec{v}=\begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix}}_{\text{Both in } R^2 \text{ and } R^3}$

$\vec{u}+\vec{v}=\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ 0 \end{bmatrix} \longrightarrow$ Still in R^3 and R^2

- Closed under Scalar multiplication: If multiplying vector by scalar vector should remain in same subspace

Ex) $\vec{v}=\begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow R^2$ $\quad \vec{u}=2 \quad$ $\vec{u}\times\vec{v}=\begin{bmatrix} 2 \\ 4 \end{bmatrix} \rightarrow$ still R^2

- Subspace containing all solutions to $A\vec{x}=0$

Example: $x_1-4x_2-3x_3-7x_4=0$

$2x_1-x_2+x_3+7x_4=0$

$x_1+2x_2+3x_3+11x_4=0$

$$\underbrace{\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}}_{A} \underset{\vec{x}}{\vec{x}} = \underset{\vec{0}}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 0 & 1 & 5 & 0 \\ 0 & \boxed{1} & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Reduced Echelon

2 pivots, 4 variables so $4-2=2$ free variables

Choose $x_4=p$, $x_3=s$

Row 2: $x_2+x_3+3x_4=0 \longrightarrow x_2=-s-3t$

Row 1: $x_1+x_3+5x_4=0 \longrightarrow x_1=-s-5t$

$$\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} -s-5t \\ -s-3t \\ s \\ t \end{bmatrix}$$

$$\underbrace{=s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

So subspace where $A\vec{x}=\vec{0}$ contains all liner combinations of $\underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$

2D-space b/c use 2 vectors to form linear combos

4.3 Linear Combinations and Linear Independence

- Linear combination: $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$
- Linearly independent: $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$

Linearly independent if # column vectors matches # pivots

Example: $\vec{v}_1 = \begin{bmatrix}1\\3\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}2\\4\end{bmatrix}$

$$\underbrace{\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix} = \begin{bmatrix}1&2\\3&4\end{bmatrix} \longrightarrow \begin{bmatrix}\boxed{1}&2\\0&\boxed{-2}\end{bmatrix}}$$

Two pivots, two vectors, linearly indp.

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \longrightarrow \begin{bmatrix}1&2&0\\3&4&0\end{bmatrix} \longrightarrow \begin{bmatrix}1&2&0\\0&-2&0\end{bmatrix}$$

Example: $\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\vec{v}_4 = \begin{bmatrix}1\\2\\3\end{bmatrix}$

$$\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4\end{bmatrix} = \begin{bmatrix}1&0&0&1\\0&1&0&2\\1&0&1&3\end{bmatrix} \longrightarrow \begin{bmatrix}\boxed{1}&0&0&1\\0&\boxed{1}&0&2\\0&0&\boxed{1}&2\end{bmatrix}$$

Not linearly indp.

Example: $\vec{v}_1 = \begin{bmatrix}-1\\-17\\-3\\9\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}14\\7\\2\\-2\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}15\\5\\1\\-2\end{bmatrix}$ Linearly independent?

3, 4-component vectors → may or may not be independent

$$\begin{bmatrix}-1&14&15\\-17&7&5\\-3&2&1\\9&-2&-2\end{bmatrix}$$

Not square so can't find determinant

Reduce to count pivots $\longrightarrow \begin{bmatrix}\boxed{-1}&14&15\\0&\boxed{-231}&-250\\0&0&\boxed{1}\\0&0&0\end{bmatrix}$ 3 pivots 3 vectors $\boxed{\text{Linearly independent}}$

Are there free variables?

Solving: $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

$$\begin{bmatrix}-1&14&15&0\\-17&7&5&0\\-3&2&1&0\\9&-2&-2&0\end{bmatrix} \longrightarrow \begin{bmatrix}-1&14&15&0\\0&-231&-250&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

Zero row: variable <u>Without</u> pivots in their column are free

Here we have pivot in 1st, 2nd, 3rd column so c_1, c_2, c_3 are NOT free

Span

- We say vectors $\vec{i} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{j} = \begin{bmatrix}0\\1\end{bmatrix}$ span R^2 b/c we can make every possible R^2 vector with linear combos of $\vec{i}$ and $\vec{j}$
  - → We can fill up R^2 using $\vec{i}$ and $\vec{j}$
- We write span $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = R_2$

  likewise, span $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = R_3$

Example:

$x_1 - 3x_2 + x_3 + x_4 = 0$

$x_1 + 2x_2 + x_3 + 11x_4 = 0$

$x_1 + x_2 + x_3 + 9x_4 = 0$

$$\begin{bmatrix}1&-3&1&1\\1&2&1&11\\1&1&1&9\end{bmatrix} \longrightarrow \begin{bmatrix}1&-3&1&1\\0&1&0&2\\0&1&0&2\end{bmatrix} \longrightarrow \begin{bmatrix}\boxed{1}&-3&1&1\\0&\boxed{1}&0&2\\0&0&0&0\end{bmatrix}$$

Two pivots so 2 free variables

$x_1 - 3x_2 + x_3 + x_4 = 0$

$x_2 + 2x_4 = 0$

Let $x_4 = t$ $x_3 = s$

$x_4 = t$

$x_3 = s$

$x_2 = -2t$

$x_1 = -7t - s$

$$X = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-7t-s\\-2t\\s\\t\end{bmatrix} = \underbrace{\begin{bmatrix}-7\\-2\\0\\1\end{bmatrix}t + \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}s}_{\text{Solution space spanned by these two vectors}}$$

4.4 Bases and Dimensions of Vector Space

For $\mathbb{R}^n$ you need at least n vectors, but n vectors may not be enough

- Exactly n if linearly independent
- More than n if not linearly independent

Basis: linearly indp. vectors

Example: Solution space of $x_1 - 2x_2 - 5x_3 = 0$

$2x_1 - 3x_2 - 13x_3 = 0$

Solution space: Contains ALL Solutions $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Solve: $\begin{bmatrix} 1 & -2 & -5 & 0 \\ 2 & -3 & -13 & 0 \end{bmatrix} \longrightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \boxed{1} & -2 & -5 & 0 \\ 0 & \boxed{1} & -3 & 0 \end{array}$

Column 3 no pivot
$\therefore$ $x_3 = r$ (free variable)

Row 2: $x_2 - 3x_3 = 0$

$x_2 = 3x_3 = 3r$

Row 1: $x_1 - 2x_3 - 5x_3 = 0$

$x_1 = 2x_2 + 5x_3$

$= 6r + 5r = 11r$

$$\boxed{\text{so} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11r \\ 3r \\ r \end{bmatrix} = r \begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}}$$

Solution space has basis $\begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}$ and is one-dimensional

Choosing x_1 or x_2 to be free gives alternative base

4.5 Row and Column Spaces

To find basis for Row(A), perform row op to get echelon matrix, then pivot rows

To find basis for Col(A), perform row operations and identify pivot columns in echelon form, then the corresponding columns in the original matrix form a basis for Col(A)

Example: $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 5 & -9 & 10 \\ 1 & 6 & -29 & -11 \end{bmatrix}$ Find basis for Row(A) and Col(A)

$\xrightarrow{\text{Row op}} \begin{bmatrix} \underline{1} & 1 & 1 & 1 \\ 0 & \underline{2} & -12 & 7 \\ 0 & 0 & 0 & \underline{-59} \end{bmatrix}$

Pivot rows: all three so basis for Row(A) are the pivot rows of echelon form: $\{[1\ 1\ 1\ 1], [0\ 2\ -12\ 7], [0\ 0\ 0\ -59]\}$

Pivot columns: 1, 2, 4 so basis for col(A) are corresponding columns of A:

$$\underbrace{\left\{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ -11 \end{bmatrix}\right\}}_{\text{Same as identifying which of the vectors that form columns of A are linearly indp}}$$

5.1 Intro to Linear Second Order Equations

Wronskian: $W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$ if $\neq 0$ linearly indp

Characteristic polynomials

$y' = r, \quad y'' = r^2$, etc.

Example: $y'' + 3y' + 2y = 0$

Characteristic eq: $r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r_1 = -2 \quad r_2 = -1$

Solutions: $y_1 = e^{r_1 x} = e^{-2x}$

$y_2 = e^{r_2 x} = e^{-x}$

General Solution: $\boxed{y = C_1 e^{-2x} + C_2 e^{-x}}$

This is the case when r's are real and distinct

Example: Repeated Roots Case

$4y'' + 4y' + y = 0$

Characteristic eq: $4r^2 + 4r + 1 = 0$ (if you forget, just plug $y = e^{rx}$ into equation, the characteristic eq. will result)

$(2r+1)(2r+1) = 0$

$r_1 = -\frac{1}{2} \quad r_2 = -\frac{1}{2}$ repeated

Form y_1 as usual: $y_1 = e^{r_1 x} = e^{-1/2x}$

General Solution: $y = C_1 e^{-1/2x} + C_2 x e^{-1/2x}$

5.2 General Solutions of Linear Equations

n^{th}-order linear: $y^{(n)} + \underline{p_1(x)} y^{(n-1)} + \underline{p_2(x)} y^{(n-2)} + \dots + \underline{P_{n-1}(x)} y' + \underline{P_n(x)} y = \underline{F(x)}$

Can't contain y

Ex) $y''' + x^2 y'' + e^x y' + 3y = \cos(x)$

Principle of Superposition: The n^{th}-order homogeneous eq. has n linearly indp solutions

$y_1, y_2, y_3, \dots, y_n$

- The linear combination of them → general solution $y = C_1 y_1 + C_2 y_2 + C_n y_n$

For higher number of functions, the Wronskian works better

$$W = \begin{vmatrix} f_1 & f_2 & \dots, & f_n \\ f_1' & f_2' & & f_n' \\ f_1'' & f_2'' & & f_n'' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix}$$

If $W \neq 0$ for an interval, then the functions are linearly independent

If $W = 0$ for all x on an interval, then the functions are linearly dependent on that interval

For example: $f_1 = 1, \quad f_2 = x, \quad f_3 = x^2$

$$W = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$$\det W = 1 \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} + 0 \begin{vmatrix} x & x^2 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2 \neq 0 \text{ for Any } x$$

so f_1, f_2, f_3 are independent
for all x's → on $(-\infty, \infty)$

Reduction of Order

Given $y'' + p(x)y' + q(x)y = 0$

If we know (somehow) one solution $\longrightarrow y_1$

The second solution can be founded by $y_2 = V(x)y_1$

Finding $v(x) \longrightarrow$ finding y_2

Example: $x^2y'' + xy' - 9y = 0$ (Euler's equation ⟶ non constant coefficients)

$y_1 = x^3$ find y_2

Assume $y_2 = Vy_1 = Vx^3$ plug into the equation

$$y_2' = 3vx^2 + v'x^3$$

$$y_2'' = 6vx + 6v'x + v''x^3$$

$$x^2(6vx + 6v'x^2 + v''x^3) + x(3vx^2 + v'x^3) - 9(vx^3) = 0$$

$$\cancel{6vx^3} + 6v'x^4 + v''x^5 + \cancel{3vx^3} + v'x^4 - \cancel{9vx^3} = 0$$

$$x^5V'' + 7v'x = 0$$

Rewrite: $V'' = -7v'x^{-1}$

$\frac{d(v')}{dx} = \frac{-7(v')}{x}$ separable in v' and x

$\frac{1}{v'}d(v') = -\frac{7}{x}dx \longrightarrow V' = Cx^{-7}$ $y_2 = Vy_1$ need v

$$V = \int Cx^{-7}dx$$

$V = \frac{-c}{6}x^{-6} + D$ Choose ANY C, D that's convenient (except those that lead to $V=0$)

Choose $C = -6$, $D = 0$: $V = x^{-6}$

So $y_2 = Vy_1 = x^{-6}x^3 = \boxed{x^{-3}}$

5.3 Homogeneous Equations With Constant Coefficients

As long as the coefficients of the linear equation are constant we can always assume solutions of the form $y = e^{rx}$

$$\left.\begin{array}{l} y'' + 5y' - 2y = 0 \\ y''' + 10y'' - 5y' + 17 = 0 \\ y^{(4)} - 100y = 0 \end{array}\right\} y = e^{rx} \text{ are the solutions}$$

2nd order: $ay'' + by' + cy = 0 \longrightarrow$ characteristic eq. $\underbrace{ar^2 + br + c = 0}$

two roots: r_1, r_2
(real and distinct, repeated, or complex)

3rd order: $ay''' + by'' + cy' + dy = 0 \longrightarrow \underbrace{ar^3 + br^2 + cr + d = 0}$

Three roots: r_1, r_2, r_3
(repeated real, distinct real)

n^{th} order: n roots Solutions are $y_1 = e^{r_1x}$, $y_2 = e^{r_2x}$, …, $y_n = e^{r_nx}$

Example: $2y'' - 3y' = 0$

$2r^2 - 3r = 0 \longrightarrow r(2r-3) = 0 \longrightarrow r_1 = 0,\ r_2 = 3/2$ (distinct)

Solutions: $y_1 = e^{r_1x}$ so $y_1 = 1$

$y_2 = e^{r_2x}$ so $y_2 = e^{3/2x}$

$\boxed{\text{General solution: } y = c_1 + c_2e^{3/2x}}$

Example: $4y'' - 12y' + 9y = 0$

$4r^2 - 12r + 9 = 0$

$(2r - 3)(2r - 3) = 0$ $r_1 = 3/2$, $r_2 = 3/2$ (repeated)

$y_1 = e^{r_1x} = e^{3/2x}$

$y_2 = xe^{r_2} = xe^{r_1} = xe^{3/2x}$

$\boxed{\text{General Solution: } y = C_1e^{3/2x} + C_2xe^{3/2x}}$

Example: Factoring cubics: $y''' + 5y'' - 100y' - 500y = 0$

$$\underline{\underline{1}}r^3 + \underline{\underline{5}}r^2 - \underline{\underline{100}}r - \underline{\underline{500}} = 0$$

Lucky situation: ratio between 1st, 2nd, and 3rd, 4th is the same (2nd is 5 times 1st, 3rd is 5 times 4th)

$$\to r^2(r+5) - 100(r+5) = 0$$

$$(r+5)(r^2-100) = 0$$

$$r_1 = -5, \quad r_2 = 10, \quad r_3 = -10$$

$$y = C_1e^{-5x} + C_2e^{10x} + C_3e^{-10x}$$

If coefficients are not patterned like above, usually guess and check and do long division

Euler's Formula: $e^{ix} = \cos x + i\sin x$

$$e^{i(bx)} = \cos bx + i\sin bx$$

$$e^{(a+bi)x} = e^{ax}e^{i(bx)} = e^{ax}(\cos bx + i\sin bx)$$

- If $r = a+bi$ is a root of the characteristic eq. then so is $r = a-bi$ (complex roots always appear in conjugate pairs)

Example: $y'' + 100y = 0$

$r^2 + 100 = 0 \to r^2 = -100 \quad r_1 = 10i \quad r_2 = -10i$

$y_1 = e^{r_1x} = e^{10ix} = e^{i(10x)}$

Note: $e^{ix} = \cos x + i\sin x$

$\boxed{y_1 = \cos 10x + i\sin(10x)}$

$y_2 = e^{r_2x} = e^{-10ix} = e^{i(-10x)}$

$y_2 = \cos(-10x) + i\sin(-10x)$

$\boxed{y_2 = \cos(10x) - i\sin(10x)}$

solutions have same real part ($\cos(10x)$) and same imaginary part ($\sin(10x)$) but opposite in sign → conjugate pairs

General solution: $y = C_1[\cos x + i\sin x] + C_2[\cos x - i\sin x]$

$y = C_1(\cos(10x) + i\sin(10x)) + C_2(\cos(10x) - i\sin(10x))$ General solution is <u>real</u> but looks complex b/c C's are also complex

$$y = \underbrace{(C_1 + C_2)}_{C_1}\cos(10x) + i\underbrace{(C_1 - C_2)}_{C_2}\sin(10x)$$

$$\boxed{y = C_1\underbrace{\cos(10x)}_{\text{real prt. of } y_1 \text{ or } y_2} + C_2\underbrace{\sin(10x)}_{\text{imag. part of } y_1 \text{ or } y_2}}$$

Example: $6y^{(4)} + 11y'' + 4y = 0$

$6r^4 + 11r^2 + 4 = 0$

$6u^2 + 11u + 4 = 0 \qquad u = r^2$

$$u = \frac{-11 \pm \sqrt{121-96}}{12} = \frac{-11 \pm 5}{12} = -\frac{4}{3}, -\frac{1}{2}$$

$$r = \underbrace{\frac{2}{\sqrt{3}}i - \frac{2}{\sqrt{3}}i}_{\text{Solve solutions as in prev. example}}, \underbrace{\frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}}i}_{\text{see prev. example}}$$

real part: $\cos\left(\frac{2}{\sqrt{3}}x\right)$

imag: $\sin\left(\frac{2}{\sqrt{3}}x\right)$

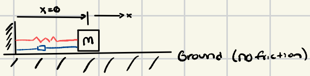
real part: $\cos\left(\frac{1}{\sqrt{2}}x\right)$

imag. part: $\sin\left(\frac{1}{\sqrt{2}}x\right)$

$$\boxed{\text{General Solution: } y = C_1\cos\left(\frac{2}{\sqrt{3}}x\right) + C_2\sin\left(\frac{2}{\sqrt{3}}x\right) + C_3\cos\left(\frac{1}{\sqrt{2}}x\right) + C_4\sin\left(\frac{1}{\sqrt{2}}x\right)}$$

5.4 Mechanical Vibration

- Mass - Spring - damper



- Spring wants to restore x to equilibrium
  - provides force of $F_s = -Kx$
- Damper resists velocity $F_d = -Cx'$

$\boxed{Mx'' + Cx' + Kx = 0}$ 2nd order linear constant coefficient eq.

M : mass
K : spring constant
C : Damper / damping

Example : mass 8 kg, no damper. Spring such that force of 40 N stretches it by 5cm.

Solve for mass position if $x(0) = 0$, $x'(0) = 10$ m/s

$Mx'' + Cx' + Kx = 0$ $M = 8$ $C = 0$ $K =$ to be found

Hooke's Law: $F = Kx$ ← change from equilibrium

$40 = K(0.5)$ ← 5cm in m

$K = 800$

→ $8x'' + 800x = 0$

$r^2 + 100 = 0$ $r = \pm 10i$

$\boxed{x(t) = C_1 \cos(10t) + C_2 \sin(10t)}$

$x(0) = 0$, $x'(0) = 10$

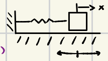
$x'(t) = -10C_1 \sin(t) + C_2 \cos(10t)$

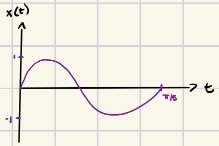
$x(0) = 0 \longrightarrow 0 = C_1 \cos(0) + C_2 \sin(0) = C_1$

$x'(0) = 10 = -10C_1 \sin(0) + 10C_2 \cos(0) = 10C_2 \longrightarrow C_2 = 1$

$\boxed{x(t) = \sin(10t)}$

circular freq. (rad/s)





Period: $\frac{2\pi}{10} = \frac{\pi}{5}$ rad

Circ. freq. : 10 rad/s

Amplitude : 1

Frequency: $\frac{1}{\text{period}} = \frac{5}{\pi}$ Hz (# cycles per second)

Same setup but now $x(0) = 2$ ($x'(0) = 10$ as before)

$x(t) = c_1 \cos(10t) + c_2 \sin(10t)$

$x'(t) = -10c_1 \sin(10t) + 10c_2 \cos(10t)$

$x(0) = 2 \longrightarrow \ldots \longrightarrow c_1 = 2$

$x'(0) = 10 \longrightarrow \ldots \longrightarrow c_2 = 1$

Now $\boxed{x(t) = 2\cos(10t) + \sin(10t)}$ Amplitude?

Alternate form: $x(t) = C\cos(\overset{\text{omega}}{\omega} t - \overset{\text{phase shift}}{\delta})$ $\longrightarrow$ $x(t) = A\cos(\omega t)B\sin(\omega t) = C\cos(\omega t - \delta)$

ω: circular frequency

δ: phase shift

Using: $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b) \longrightarrow C = \sqrt{A^2+B^2}$

$\delta = \tan^{-1}\left(\frac{B}{A}\right)$

↳ Dangerous since doesn't tell quadrant refer to A_x and B_y to determine

- Inverse tan gives Q IV and Q I

$x(t) = \overset{A}{2}\cos(10t) + \overset{B}{1}\sin(10t)$

$\boxed{x(t) = \underset{C}{\sqrt{5}} \cos(10t - \underset{\delta}{0.464})}$ Amplitude $= \sqrt{5}$

Back to $mx'' + cx' + kx = 0 \longrightarrow m, c, k \neq 0$

$mr^2 + cr + k = 0$

$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$ $c^2 - 4km$ (discriminant) determines type of roots

Not units, k and m of problem

- If $c^2 - 4km > 0$ ($c^2 > 4km$) roots are real and distinct r_1, r_2

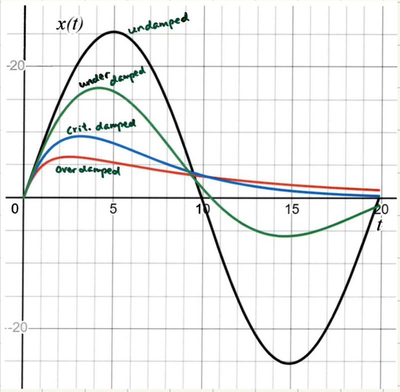
  $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ Strong damper ($c^2 > 4km$) $\longrightarrow$ Over damped (Stronger than needs to be)

- If $c^2 - 4km = 0$ ($c^2 = 4km$) roots are real and repeated $r = r_1 = r_2$ $\longrightarrow x(t) = c_1 e^{rt} + c_2 t e^{rt}$ Critically damped (tuned just right)

- If $c^2 - 4km < 0$ ($c^2 < 4km$) roots are complex $r = a + bi$

  $x(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$ Under damped (weak)

  only case with oscillation



Stronger the damper the higher the amplitude