MA 262 exam 2 review

ROW AND COLUMN SPACES

- row space: all possible linear combos of the rows of matrix
  (space spanned by its rows
- column space: all possible lin. combos of columns
  (space spanned by columns

STEPS:
- use Gaussian elimination to find pivots
  ↳ produces row-equivalent matrices - preserves row space
- rows of new matrix containing pivots create row space
- columns of new matrix with pivots correspond w/ cols in OG matrix that compose column space

NOTE: $col(A) = row(A^T)$

MISCELLANEOUS

* basis for subspace = column space (or basis for span)

* solution space = null space = all possible solutions

- Determinant prop:
  - $\det(AB) = \det(A)\det(B)$ $\quad \det(A^{-1}) = \frac{1}{\det(A)}$ $\quad$ square matrices: $\det(-A) = -\det(A)$
  - $\det(CA) = C^n \cdot \det(A)$ $\quad \det(A^T) = \det(A)$
    where n = # col in A
- a_{ij} of $A^{-1} = \frac{\text{cofactor } a_{ji} \text{ of } A}{\det A}$ → (follow sign rule, don't include a_{ji}, find matrix like cofactoring)
- $0 \neq n \Rightarrow$ no solutions
  $0 = 0 \Rightarrow$ infinite solutions

* basis for null space ⇒ solve for $A\vec{x} = 0$ and put $\vec{x}$ in vector form
  ↳ dim(null space): # col w/o pivots → Nullity(A)

- rank(A): # of col w/ pivots
- 1 vector in "span" of 2 others: $A = c_1 B + c_2 C$ (A; B, C)
- nonsingular matrix: square matrix whose det ≠ 0 (invertible)

POPULATION MODELS

$$\frac{dP}{dt} = [\beta - \delta]P \Longrightarrow \underbrace{aP}_{\text{birth rate}} - \underbrace{bP^2}_{\text{death rate}}$$

(β: birth rate, δ: death rate)

usually have to solve for a, b

* review partial fractions for applications

most common form:

$$\frac{dP}{dt} = kP(M - P)$$

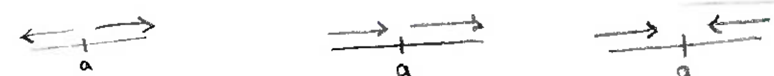
(M: limiting capacity)

$\begin{cases} 0 < P < M & \text{(growth)} \\ P > M & \text{(decline)} \end{cases}$

EQUILIBRIUM / STABLE SOLUTIONS

- Critical points ⇒ equilibrium solution

* phase charts (like slope fields) help us analyze these



unstable equilibrium - go away from eq. point

semistable equil. - one side approaches, other side goes away

stable equilibrium - both sides want to return to eq. point

EULER'S METHOD

- helps us approx. when exact solution cannot be found

$$y_{n+1} = y_n + f(x_n, y_n) \cdot h$$
$$x_{n+1} = x_n + h$$

- iterate until x_{n+1} = target x
- smaller step size, h, = more accurate estimate

SOLVING SYSTEMS - MATRICES

- elimination → use row operations:
  - swap any two rows
  - multiply any row by nonzero constant
  - add one row to another

* Gaussian elimination: process of making a matrix in row-echelon form

row-echelon form: solution is buried

- "pivot" is below and to the right of previous pivot
- all numbers below pivot are zero
- column w/o pivot = free variable (infinitely many solutions)
- row of zeroes at bottom

(if not possible, no solution!)

ex. $\begin{bmatrix} \boxed{1} & -1 & 1 & 7 \\ 0 & \boxed{1} & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad x_3 = t$, $x_2 = 3t - 2$, $x_1 = 2t + 5$

reduced-row echelon form: solution is explicit

- every pivot is the only nonzero number in column
  - ↳ every pivot is 1
- unique for each matrix
- applied for homogeneous systems (right-most column is all zeros)

* Gauss-Jordan elimination: process to go to RREF form

MATRIX OPERATIONS

- matrix addition - add two matrices of same dimension (same # row / # col)
- matrix subtraction - subtract " "
  (given an equation in a square, add it to itself and check if conditions are met)
- scalar multiplication - c × each element in A
- matrix multiplication - only possible if # col A = # row B
  - ↳ AB ≠ BA - order matters! (except for identity matrix)

$R_A \times \underbrace{C_A \cdot R_B}_{\text{must be equal}} \times C_B$ ⟹ dimension of resulting matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

- matrix inverse:
  - ↳ for 2×2: $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ swap main diagonal, change signs of off-diagonal
    - if det=0, NOT INVERTIBLE
  - ↳ for n×n: $[A \mid I] \xrightarrow{\text{row op.}} \cdots \rightarrow [I \mid A^{-1}]$ (nxn identity)
  - ↳ NOTE: in matrix form $A\vec{x} = B$, $\vec{x} = A^{-1}b$

DETERMINANTS

- cofactor expansion method:
  - choose row/col to expand on (most amt of zeros) in matrix A
  - block out row and col of a_{ij}, create new matrix w/ remaining row/col
  - mult. det of new matrix to a_{ij} accdg to sign pattern (think checkerboard) $\begin{smallmatrix} + & - & + \\ - & + & - \\ + & - & + \end{smallmatrix}$
  - repeat for all elements in chosen row/col → sum = det A

CRAMER'S RULE - another way to solve system (along w/ Inverse, RREF)

for matrix equation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad x_1 = \frac{\begin{vmatrix} b_1 & a_{21} \\ b_2 & a_{22} \end{vmatrix}}{\det A} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{12} & b_2 \end{vmatrix}}{\det A}$$

* same idea for larger matrices

* DETERMINANTS HACK: special matrices

- transpose matrix: col of A becomes row of A^T, row of A becomes col of A^T > $\det(A) = \det(A^T)$
- triangular matrix: everything above OR below main diagonal is all zeroes
  - det A = product of #s on main diagonal

VECTOR SPACE $\mathbb{R}^n$ + SUBSPACES

span of $\mathbb{R}^n$: space covered by a set of vectors

- contains vectors w/ n components that can be operated upon
- vector spaces have 8 properties:

$\vec{u}+\vec{v} = \vec{v}+\vec{u}$ $\quad \vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ $\quad \vec{u}+(-\vec{u}) = (-\vec{u})+(\vec{u})$

$a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$ $\quad (a+b)\vec{u} = a\vec{u}+b\vec{u}$ $\quad 1\cdot(\vec{u}) = \vec{u} = 0+\vec{u}$

- SUBSPACE is a part of vector space, has 3 properties:
  - closed under addition
  - closed under scalar multiplication } resulting vector should still be in "bounds" of vector space
  - has zero vector

* null space: all possible solutions to $A\vec{x} = 0$ (composed of vectors) — also called "solution space"
  - ↳ transform into augmented matrix, solve for $x_1, x_2 \dots x_n$
  - for lin. ind. vectors, null space only contains zero vector

LINEAR INDEPENDENCE VS DEPENDENCE

- Lin ind. ⟹ $\underbrace{c_1v_1 + c_2v_2 + \dots + c_nv_n = 0}_{\text{linear combination}}$, where $c_1 = c_2 = \dots = c_n = 0$
- Lin dep. ⟹ one vector is a multiple / duplicate of another

TO DETERMINE INDEPENDENCE OF VECTORS:

- # vectors > # components ⟹ LIN DEP
- # vectors ≤ # components ⟹ test

* create matrix of vectors, solve det:
  - det = 0 ⟹ LIN DEP
  - det ≠ 0 ⟹ LIN IND

or...

- create matrix and find REF - if # pivots = # vectors — LIN IND

BASES & DIMENSIONS OF VECTOR SPACE

- basis: smallest spanning set (n lin. ind. vectors) — not unique
  - ↳ bases: vectors in basis
- dimension: # of vectors in a vector spaces base