line integrals

scalar line integral

in the form $\int_c f(x,y)\,ds$ w/ curve c [SCALAR]

↓

- parameterize curve if it is not already
- replace x,y components with respective parameterized ones

↓

- take derivative of parameterized curve
- find magnitude of this derivative

↓

- use bounds ($t \in [a,b]$) from parameterization as bounds for the integral

↓

- repeat for each curve

↓

example

$\int (y-z)\,ds$ C: helix; $0 \le t \le 2\pi$ $\langle 10\cos t, 10\sin t, t\rangle$

$r'(t) = \langle -10\sin t, 10\cos t, 1\rangle$

$|r'(t)| = \sqrt{100\underbrace{(\sin^2 t+\cos^2 t)}_{1}+1^2}$

$|r'(t)| = \sqrt{101}$

$\int_0^{2\pi} (10\sin t - t)\sqrt{101}\,dt$

$\sqrt{101}\left[-10\cos t - \frac{t^2}{2}\right]_0^{2\pi}$

$= \sqrt{101}\left[(-10-2\pi^2)-(-10)\right]$

$= -2\pi^2\sqrt{101}$

tangent vector line integral

in the form $\int_c \vec{F}\cdot\vec{T}\,ds$ w/ curve c [vector]

circulation, curl, work, flow

↓

$\int_c \vec{F}\cdot\vec{T}\,ds$ becomes $\int_c \vec{F}\cdot d\vec{r}$

→ $ds = r'(t)\,dt$

↓

- parameterize curve if it is not already
- replace components w/ respective parameterized components

↓

- take derivative of parameterized curve
- dot derivative w/ $\vec{F}$

↓

- use bounds ($t \in [a,b]$) from parameterized curve as bounds for integral

↓

example

$\int_c \vec{F}\cdot\vec{T}\,ds$ C: $\langle 14t, 7t^2\rangle$ $0 \le t \le 1$

F: $\langle x, y\rangle$

$r'(t) = \langle 14, 14t\rangle$

$F = \langle 14t, 7t^2\rangle$

$\int_0^1 \vec{F}\cdot r'(t)\,dt = \int_0^1 196t + 98t^3\,dt$

$= 98t^2 + \frac{49}{2}t^4 \Big|_0^1$

$= 98 + \frac{49}{2} = \frac{245}{2}$

→

OR $\oint_c f\,dx + g\,dy$

↳ changes to $\iint_R g_x - f_y\,dA$

↓

$\langle f, g\rangle$

- take partial derivatives ie. $\frac{\partial}{\partial x}(g)$ and $\frac{\partial}{\partial y}(f)$

↓

- find bounds of the region

★ draw the region ★

↓

- set up the double integral

★ must be a closed loop & positive ccw

↓

example

what is the circulation?

$C_1: \langle a\cos t, a\sin t\rangle$ $0 \le t \le \pi$

$C_2: \langle t, 0\rangle$ $-a \le t \le a$

$F = \langle -4x, -4y\rangle$

$g_x = 0,\ f_y = 0$ } conservative

$\int_0^\pi \int_0^a (0-0)\,\underbrace{r\,dr\,d\theta}_{dA}$

$= 0$

normal vector line integral

flux, normal, divergence

$\oint_c f\,dy - g\,dx$

↳ changes to $\iint_R f_x + g_y\,dA$

↓

$\langle f, g\rangle$

- take partial derivatives ie $\frac{d}{dx}(f)$ and $\frac{d}{dy}(g)$

↓

- find bounds of the region

★ draw the region ★

↓

- set up the double integral

★ must be a closed loop & positive ccw

↓

example

what is the flux

$C_1: \langle a\cos t, a\sin t\rangle$ $0 \le t \le \pi$

$C_2: \langle t, 0\rangle$ $-a \le t \le a$

$F = \langle -4x, -4y\rangle$

$f_x = -4,\ g_y = -4$ } non-cons.

$\int_0^\pi \int_0^a (-4-4)\,r\,dr\,d\theta$

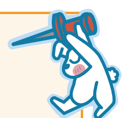
$= \int_0^\pi \left(-4r^2\right|_0^a$

$= \int_0^\pi -4a^2\,d\theta$

$= -4a^2\theta\Big|_0^\pi = -4\pi a^2$

green's

how to parameterize:



line: (x and y) → one parameter t

surface: (x, y, and z) → two parameters u and v

for planes (surface) let $x = u,\ y = v$ then $z = z(u,v)$ ↳ replace x and y with u and v	for cylinders (surface) let $u = \theta,\ z = v$ then $x = r\cos\theta,\ y = r\sin\theta,\ z = v$ constant
for paraboloids/spheres (surface) let $r = u,\ \theta = v$ then $x = u\cos\theta,\ y = u\sin\theta$ ↳ $z = z(u,v)$ → replace x and y	circles (line) $\langle r\cos t, r\sin t\rangle$ constant line: $r_0 + t\vec{r}$

surface integrals

is the provided field scalar?

↓

is the field good in cartesian? and maybe polar...

→ (if not)

- parameterize the surface in terms of u and v

↓

- take the partial w/ respect to u and v of the parameterization "r"

↓

- cross $\vec{r}_u$ and $\vec{r}_v$
$|\vec{r}_u \times \vec{r}_v|$

↓

- plug in parameterized x and y values (not the derivative of the parameterization!) in the integral func.
$\iint \underbrace{g(x,y,z)}_{\text{this one}}$

↓

- set up the integral
→ replace dS with $|\vec{r}_u \times \vec{r}_v|$

↓

example

surface area of cone

coor. system: polar/cylindrical

$z^2 = 9(x^2+y^2) \quad 0 \le z \le 9$

$r(t) = \langle u\cos v, u\sin v, 3u \rangle$

$0 \le u \le 3 \quad 0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos v, \sin v, 3 \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$\vec{r}_u \times \vec{r}_v = -3u\cos v\,\hat{i} + 3u\sin v\,\hat{j} + u\,(\cos^2 v + \sin^2 v)$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{9u^2(\cos^2 v + \sin^2 v) + u^2}$

$= u\sqrt{10}$

$\int_0^{2\pi}\int_0^3 u\sqrt{10}\,du\,dv = \frac{u^2}{2}\sqrt{10}\Big|_0^3$

$= \int_0^{2\pi} \frac{9}{2}\sqrt{10}\,dv = 9\pi\sqrt{10}$

↓ yes (field good in cartesian)

try the explicit form!
careful, this does not adjust for cylindrical and spherical coordinates

↓

$dS = \sqrt{1 + z_x^2 + z_y^2}\,dA$
- put equation in form $z = \dots$

↓

- take partial with respect to x and y
- plug into dS eq. (the sqrt thing)

↓

- fix coordinate system if needed

↓

- replace dS w/ dS solved above in
$\iint g(x,y,z)\,dS$

↓

example

Surface area of cone

coor. system: polar/cylindrical

$z^2 = 9(x^2+y^2) \quad 0 \le z \le 9$

$z = 3\sqrt{(x^2+y^2)}$

$dS = \sqrt{1^2 + \left(\frac{3x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{3y}{\sqrt{x^2+y^2}}\right)^2}$

$= \sqrt{1 + \frac{9(\cancel{x^2+y^2})}{\cancel{x^2+y^2}}} = \sqrt{10}$

change to polar

$0 \le \theta \le 2\pi$

$9^2 \rightarrow \quad z^2 = 9r^2$

$81 = 9r^2$

$r = 3$

$0^2 \rightarrow \quad 0 = 9r^2$

$r = 0$

$\therefore 0 \le r \le 3$

set up integral

$\int_0^{2\pi}\int_0^3 \sqrt{10}\,\underbrace{r\,dr\,d\theta}_{dV}$

solve...

$\frac{r^2}{2}\sqrt{10}\Big|_0^3 = \frac{9}{2}\sqrt{10}$

$\int_0^{2\pi} \frac{9}{2}\sqrt{10}\,d\theta$

$= 9\pi\sqrt{10}$

is the provided field a vector?

↓ yes

finding the flux of the curl?

↓ yes!

use stokes' theorem

↓

key words: curl, boundary, any surface, open

↓

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot dr$

S: surface

C: boundary curve

check if $\nabla \times \vec{F}$ (curl) is 0 → $\oint \vec{F}\cdot dr = 0$

↓ steps

- parameterize boundary curve (with parameter t)
- replace vector field components w/ parameters

↓

- take derivative of parameterization
- dot $\vec{F}$ w/ r' (the deriv.)

↓

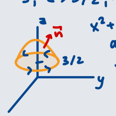
$\int_{t_1}^{t_2} \vec{F}\cdot r'(t)\,dt$
- take bounds from parameterization, integrate

↓

example

$\vec{F} = \langle y, -x, 0 \rangle$

S: part of sphere radius 3, $z > 3/2$; normal upward



$x^2 + y^2 + z^2 = 9$

at $z = 3/2$

$x^2 + y^2 + \frac{9}{4} = 9$

$x^2 + y^2 = \frac{27}{4} \therefore$ radius $= \frac{\sqrt{27}}{2}$

$r(t) = \left\langle \frac{\sqrt{27}}{2}\cos t, \frac{\sqrt{27}}{2}\sin t, 3/2 \right\rangle$

$0 \le t \le 2\pi$

$r'(t) = \left\langle -\frac{\sqrt{27}}{2}\sin t, \frac{\sqrt{27}}{2}\cos t, 0 \right\rangle$

$\vec{F} = \langle y, -x, 0 \rangle = \left\langle \frac{\sqrt{27}}{2}\sin t, -\frac{\sqrt{27}}{2}\cos t, 0 \right\rangle$

$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(-\frac{27}{4}\sin^2 t - \frac{27}{4}\cos^2 t\right)dt$

$= -\frac{27}{4}\int_0^{2\pi} dt = \frac{-27}{4}\cdot 2\pi = -\frac{27}{2}\pi$

→ no (not finding the flux of the curl)

ensure the shape is closed! and a defined vector field throughout

key words: divergence

surface + plane, full sphere

↓

use divergence theorem

↓

$\iiint_E \text{div}\,\vec{F}\,dV = \iint_S \vec{F}\cdot\vec{n}\,dS$

E: enclosed volume

S: surface bounding the volume

↓

- dot del ∇ with the field $\vec{F}$

↳ if 0 stop

↓

if $\nabla\cdot\vec{F}$ is a constant multiply this constant by the volume of the shape

↓ if not a constant

set up the triple integral of the bounding surface

★ make sure to fix dV to match coord. system

↓

example

$\vec{F} = \langle -x, y\,6z \rangle$

S: paraboloid $z = 2 - x^2 - y^2$

base on xy plane

$\nabla\cdot\vec{F} = \frac{d}{dx}(-x) + \frac{d}{dy}(y) + \frac{d}{dz}(6z)$

$= 6$

I don't know the volume of a paraboloid so...

base: $x^2 + y^2 = 2$ (at $z = 0$)

cylindrical looks good...

$0 \le r \le \sqrt{2}, \quad 0 \le \theta \le 2\pi$

and z... $0 \le z \le 2 - r^2$ ← x^2+y^2

So... $\int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^{2-r^2} 6\,r\,dz\,dr\,d\theta$

adjust for polar

$\nabla\cdot\vec{F}$

$= \int_0^{2\pi}\int_0^{\sqrt{2}} 6(2 - r^2)\,r\,dr\,d\theta$

$= \int_0^{\sqrt{2}} 12r - 6r^3\,dr = 6r^2 - \frac{3}{2}r^4\Big|_0^{\sqrt{2}}$

$= \int_0^{2\pi} (12 - 6)\,d\theta = 2\pi\cdot 6 = 12\pi$