Vector Review

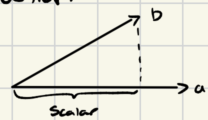
- Unit Vector : $\frac{\vec{u}}{|\vec{u}|}$ Magnitude of 1
- Circle / sphere : $(x-i)^2 + (y-j)^2 + (z-k)^2 = R^2$

  (i, j, k) : center

  R : radius

  - Complete square to get to standard form

- Projection :



$$\text{Proj}_a b = \underbrace{\frac{a \cdot b}{|a|}}_{\text{Scalar}} \cdot \frac{a}{|a|}$$

- Cross product : $\underbrace{|u \times v|}_{\text{Area of parallelogram}} = |u||v|\sin\theta$

  $\frac{1}{2}|u \times v|$: Area of triangle

- Dot product : $u \cdot v = |u||v|\cos\theta$ ← Angle between vectors

  - If $u \cdot v = 0$ vectors $\perp$

Lines and Planes

- Line equation :
  - Vector form : $\vec{r}(t) = \underbrace{\langle x_0, y_0, z_0 \rangle}_{\text{vector, origin to point}} + t \underbrace{\langle a, b, c \rangle}_{\text{Directional derivative}}$

  - Parametric form : $x = x_0 + at$ $y = y_0 + bt$ $z = z_0 + ct$

  - Parallel lines : directional derivative cross product $= 0$
  - Intersection : one parametric pair $r(t) = s(t)$

    1) Set parametric x, y, z components of the two equal

    2) Solve for s or t

    3) If equations equivalent intersections occur

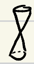
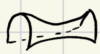
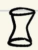
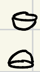
  - Collision : Time interval (s and t) are equal at intersection

- Plane equation: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$
  - Normal vector $\langle a,b,c \rangle$
  - Goes through point (x_0, y_0, z_0)
  - Two planes // if normal vectors //
  - Two planes $\perp$ if normal vectors $\perp$

  - Cross product of vectors gives normal vector

Quadratic Surfaces

- Draw traces to get idea of shape

  - $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Ellipsoid 
  - $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ Elliptic Cone 
  - $\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$ Hyperbolic Paraboloid  (like saddle)
  - $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$ Elliptic Paraboloids 
  - $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of one sheet 
  - $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Hyperboloid of two sheets 

  $x^2=y$ or $y^2=x$: Parabola
  $x^2-y^2=c$ (constant) : hyperbola

  - $x^2+y^2=1$: Cylinder radius 1, parallel to z-axis

Vector Valued Functions

- Vector value function: $r(t)=\langle x(t), y(t), z(t)\rangle$
  - Correlate x, y, z of equations to components of $r(t)$
  - Draw graph
- Finding domain: determine constraints of x, y, z components, find where they all overlap

- Unit Tangent vector: $T=\frac{r'(t)}{|r'(t)|}$

Motion in Space

- Position vector: $r(t) = \int v(t)\,dt + \langle c, c, c \rangle$
- Velocity vector: $v(t) = r'(t) = \int a(t) + \langle c, c, c \rangle$
- Acceleration: $a(t) = v'(t)$

Ex)

6. A particle has acceleration $\mathbf{a} = \langle 6t - 2, -1/t^2, 0 \rangle$. It is known that the velocity at time $t = 1$ is $\mathbf{v}(1) = \langle 1, 1, 1 \rangle$ and that the position vector at time $t = 1$ is $\mathbf{r}(1) = \langle 0, 0, 3 \rangle$. Find the magnitude of the position vector at time $t = 2$.

A. $\sqrt{16 + \ln 4}$
B. $\sqrt{16 + (\ln 2)^2}$
C. $\sqrt{32 + (\ln 2)^2}$
D. 4
E. $\sqrt{32 + (\ln 4)^2}$

$v(t) = \int a(t) = \langle 3t^2 - 2t, \frac{1}{t}, 0 \rangle + \langle c, c, c \rangle$

$v(1)_x = 3 - 2 = 1 \longrightarrow C = 0$

$v(1)_y = 1 + C = 1 \longrightarrow C = 0$

$v(1)_z = 0 + C = 1 \longrightarrow C = 1$

$v(t) = \langle 3t^2 - 2t, \frac{1}{t}, 1 \rangle$

$r'(t) = \langle t^3 - t^2, \ln(t), t \rangle + \langle c, c, c \rangle$

$C_x = 0 \quad C_y = 0 \quad C_z = 2$

$r(t) = \langle t^3 - t^2, \ln(t), t + 2 \rangle$

$r(2) = \langle 4, \ln(2), 4 \rangle$

$|r(2)| = \sqrt{32 + [\ln(2)]^2}$

Length of Curve

- Length of curve: $L = \int_a^b |r'(t)|dt$

If $|r'(t)| = 1$ arc length is parameter, change t of value to u then make uper bound t

Ex) $\vec{r}(t) = \langle t^2, 8t^2, \sqrt{79}\,t^2 \rangle$ Arc length as parameter $1 \le t \le 4$?

$r'(t) = \langle 2t, 16t, 2t\sqrt{79} \rangle$

$|v(t)| = |\vec{r}'(t)| = \sqrt{(2t)^2 + (16t)^2 + (2t\sqrt{79})^2} = \underline{24t} \ne 1$ — 1) Check if acr length is parameter

Curve does not use arc length parameter

$S(t) = \int_1^t \underline{24u} = 12u^2 \Big|_1^t = (12t^2 - 12) = S$ — 2) If ① is false change t to u, integrate to get S

$S = (12t^2 - 12) \longrightarrow S + 12 = 12t^2 \longrightarrow t = \sqrt{\frac{S+12}{12}}$ — 3) Solve for t

$\vec{r}(t) = \langle t^2, 8t^2, \sqrt{79}\,t^2 \rangle$

$\longrightarrow r(t) = \langle \frac{1}{12}(S+12), \frac{8}{12}(S+12), \frac{\sqrt{79}}{12}(S+12) \rangle$ — 4) Plug t into original r(t)

$1 \le t \le 4 \xrightarrow{\text{Plug in}} S(t) = 12t^2 - 12 \longrightarrow 0 \le S \le 180$ — 5) Plug bounds into S equation to get new bounds

- Curvature: $K = \frac{|T'|}{|r'|} = \frac{|r'' \times r'|}{|r'|^3}$

Limits and Continuity

- Steps:
  1) Test if DNE
  2) Try $x = a$, $y = b$ and $x = y$ (if same answer try 3 other wise result DNE)
  3) Manipulate equation

$\lim_{u \to 0} \frac{\sin(u)}{u} = 1$

Ex)

7. Assuming that $\lim_{(x,y)\to(0,0)} \dfrac{\sin(4x^2+8y^2)}{x^2+2y^2}$ exists, what is its value?

A. 1/4
B. −4
C. −1/4
D. 0
E. 4

X-axis : $\dfrac{\sin(4x^2)}{x^2} = \dfrac{0}{0}$ $\qquad$ $y = x$: $\dfrac{\sin(4x^2+8x^2)}{x^2+2x^2} = \dfrac{\sin(12x^2)}{3x^2}$

y-axis : $\dfrac{\sin(8y^2)}{2y^2} = \dfrac{0}{0}$

$$\frac{\sin(4(x^2+2y^2))}{x^2+2y^2} = \frac{\sin 4u}{u} = 4$$

8. If
$$\lim_{(x,y)\to(0,0)} \frac{x^4-3a(x^2+y^2)-y^4}{x^2+y^2} = 12,$$
then the number a must be equal to

A. 4
B. 6
C. 12
D. −4
E. 3

$$\frac{x^4-y^4-3a(x^2+y^2)}{x^2+y^2} = 12$$

$$\frac{\cancel{(x^2+y^2)}(x^2-y^2)}{\cancel{x^2+y^2}} - \frac{3a\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} = 12$$

$$(x^2-y^2) - 3a = 12$$
$$a = -4$$

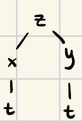
Partial Derivatives

$$\underbrace{f_{xy}}_{\text{Left to right}} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \underbrace{\frac{\partial^2 f}{\partial y\,\partial x}}_{\text{Right to left}}$$

Chain Rule

- Draw correlation tree
  - Splits mean partial derivative

Ex) $z = f(x,y) = x + y^2$ $\quad x = e^t,\ y = \ln(t)$ $\quad$ find $\underbrace{\dfrac{dz}{dt}}_{\text{Result of all paths of } t}$

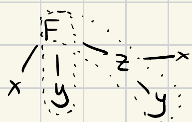


$$z_x \cdot x_t + z_y \cdot y_t = \frac{dz}{dt}$$

- If given general equation make it F

Ex) $xy + yz + xz = 3$ $\qquad$ z function of x, y $\qquad$ find $\dfrac{\partial z}{\partial y}$

$$F = xy + yz + xz - 3 = 0$$



$$F_y + F_z \cdot z_y = \emptyset = \frac{dF}{dy}$$

b/c general equation equals zero

Directional Derivative and Gradient

· Directional Derivative : $D_u f(x,y) = \overbrace{\nabla f(x,y) \cdot \vec{u}}^{\text{Dot Product}}$

unit vector specifying direction

- If given point find vector then take unit vector to get u

- Greatest increase : $\frac{\vec{\nabla} f}{|\nabla f|}$
- Greatest decrease : $-\frac{\vec{\nabla} f}{|\nabla f|}$
- Maximum rate of change : $|\nabla f|$
- $\perp$ movement : Point given $\cdot \nabla f_{(\text{point given})} = 0$

Tangent Plane / Linear Approximation

· Tangent Plane $\nabla F \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

Ex) $z = x^2 + y^2 + xy$ at $(0,1,1)$

$F = x^2 + y^2 + xy - z$

$\nabla F = \langle 2x + y, 2y + x, -1 \rangle$ $\quad \nabla F_{(0,1,1)} = \langle 1, 2, -1 \rangle$

$1(x-0) + 2(y-1) - 1(z-1) = 0$

$\boxed{x + 2y - z = 1}$

1) Find ∇F and plug in point
2) use $\nabla F \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$ using point for (x_0, y_0, z_0)
3) Solve equation

· % Change : $dz/z = \frac{f_x dx + f_y dy}{z}$

Max and Min problems

1) Find cp at $f_x = 0$ and $f_y = 0$
2) Evaluate the Discriminant : $D = f_{xx} f_{yy} - (f_{xy})^2$

$D > 0$ $\quad f_{xx} < 0$: Max

$D > 0$ $\quad f_{xx} > 0$: min

$D < 0$ $\quad$ saddle

$D = 0$ $\quad$ Inconclusive

Lagrange Multipliers

$\nabla f = \lambda \nabla g$ → constraint

1) Break into x, y, z components
2) Get components equal then solve for x, y, or z
3) Plug solve component into $g(x,y,z)$, use resulting value to find other two components
4) Plug point into $f(x,y,z)$

Ex) **1.** The extreme values of $f(x,y,z) = 3x + 2y + 6z$ with constraint $x^2 + y^2 + z^2 = 4$ are

A. The maximum of f is 7 and the minimum of f is -14
B. The maximum of f is 14 and the minimum of f is -14
C. The maximum of f is 7 and the minimum of f is -7
D. The maximum of f is 14 and the minimum of f is -7
E. The maximum of f is 28 and the minimum of f is -28

$\nabla F = \langle 3, 2, 6 \rangle \quad \nabla g = \langle 2x, 2y, 2z \rangle$

$3 = \lambda 2x \qquad 2 = \lambda 2y \qquad 6 = \lambda 2z$

$\lambda = \frac{3}{2x} \qquad \lambda = \frac{2}{2y} \qquad \lambda = \frac{6}{2z}$

$\frac{3}{2x} = \frac{2}{2y} = \frac{6}{2z} \rightarrow \underbrace{\frac{3}{2x} = \frac{1}{y}}\underbrace{\;= \frac{3}{z}}$

① $x = \frac{3y}{2}$ $\quad 3y = z$ ② → $g(x,y,z)$

$g(x,y,z) = \left(\frac{3}{2}y^2\right) + y^2 + (3y^2) = 4$

$\frac{9}{4}y^2 + y^2 + 9y^2 = 4$

$49y^2 = 16$

$y = \pm\frac{4}{7}$ (plug into ①, ②)

$x = \pm\frac{6}{7} \quad z = \pm\frac{12}{7}$

$\left(\frac{6}{7}, \frac{4}{7}, \frac{12}{7}\right) \rightarrow f(x,y,z) = 14$ max

$\left(-\frac{6}{7}, -\frac{4}{7}, -\frac{12}{7}\right) \rightarrow f(x,y,z) = -14$ min

OR

1) Set $\nabla F = \lambda \nabla G$ x, y, z components to zero and solve (ignore λ)
2) Plug into G for CP
3) Plug points into F

Ex) **7.** At which points on the curve $x^2 + y^2 = 4$ does the function $f(x,y) = 4x^2 + 10y^2$ achieve an absolute maximum?

A. $(0,2)$ and $(0,-2)$
B. $(0,2)$ only
C. $(2,0)$ and $(2,0)$
D. $(2,0)$ only
E. It does not achieve an absolute maximum

$\langle 8x, 20y \rangle = \lambda \langle 2x, 2y \rangle$

$8x = \lambda 2x \qquad 20y = \lambda 2y$

$8x - \lambda 2x = 0 \qquad 20y - \lambda 2y = 0$

$2x(8 - \lambda) = 0 \qquad 2y(10 - \lambda) = 0$

$x = 0 \qquad y = 0$

$\lambda = 8 \qquad \lambda = 10$

$y^2 = \pm 2 \qquad x = \pm 2$

$(0,2), (0,-2) \qquad (2,0), (-2,0)$

$F(0,2) = 40$
$F(0,-2) = 40$
$F(2,0) = 16$
$F(-2,0) = 16$

Integrals

- Average value: $f_{avg} = \frac{1}{A}\iint f(x,y)\,dx\,dy$ ↳ Area
- Always do a sketch

Polar Coordinates

$x^2+y^2=r^2$ $\quad dA = r\,dr\,d\theta$

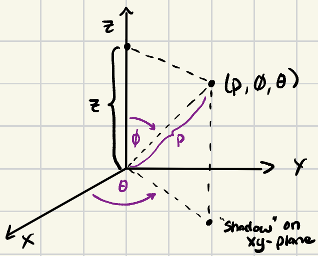
$x = r\cos\theta$

$y = r\sin\theta$

Spherical Coordinates

$\rho^2 = x^2+y^2+z^2 \qquad \rho \geq 0$

$\left.\begin{aligned} 0 \leq \theta \leq 2\pi \\ 0 \leq \emptyset \leq 2\pi \end{aligned}\right\}$ max quantities



$x = \rho\cos\theta\sin\emptyset \qquad dV = \rho^2 \sin\emptyset\, d\rho\, d\emptyset\, d\theta$

$y = \rho\sin\theta\sin\emptyset$

$z = \rho\cos\emptyset$

Integrals in Mass Calculations

- Moments:

$M_x = \iint y\, \rho(x,y)\, dA$

$M_y = \iint x\, \rho(x,y)\, dA$

- Mass: $m = \iint \underbrace{\rho(x,y)}_{\text{density}}\, dA$

- Center of mass:

X-coordinate: $\bar{x} = \frac{1}{m}\iint_R x\,\rho(x,y)\,dA$

y-coordinate: $\bar{y} = \frac{1}{m}\iint_R y\,\rho(x,y)\,dA$

z-coordinate: $\bar{z} = \frac{1}{m}\iint_R z\,\rho(x,y,z)\,dA$

- Moment of inertia: $I = \iint \rho(x,y,z)\, d^2\, dA$
  ↳ Distance from axis of rotation

Line integrals of functions and Vector fields

- Line integral: $\int_c f(x,y,z)\,ds$ C: some curve
  $ds = |r'(t)|$

Parameterize : $\vec{r}(t)$, $a \leq t \leq b$

- Work : $\int_a^b F \cdot r'(t)\, dt$

- Flux: $\int \vec{F} \cdot \vec{n}\, ds$
  - n always right of path (look at where line is going to determine if ccw or cw)
  - $n = \nabla \vec{T} = \nabla \cdot \frac{r'}{|r'|}$

Fundamental Theorem of line Integrals

- Conservative : If $\vec{F} = \langle P, Q \rangle$ $P_y = Q_x$

- Theorem: $\int_c \vec{F} \cdot dr = \phi(B) - \phi(A)$ If $\vec{F} = \vec{\nabla}\phi$

$\langle P, Q, R \rangle = \langle \phi_x, \phi_y, \phi_z \rangle$

PQ: $P_y = Q_x$
PR: $P_z = R_x$
QR: $Q_z = R_y$

$\left.\begin{array}{l} \phi_x = f(x) \\ \phi_y = f(y) \end{array}\right\}$ Integrate with respect to specific variable

$\phi =$ Sum of unique variable from integrals

$\phi \Big|_{\text{start point}}^{\text{end point}}$

Ex) **4.** Let $\mathbf{F}$ be the conservative vector field given by $\mathbf{F}(x,y) = \langle y^3+1, 3xy^2+1 \rangle$. Consider a semicircular path C_1 from $(0,0)$ to $(2,0)$, that is

$$C_1 : \{\mathbf{r}(t) = \langle 1-\cos(t), \sin(t) \rangle;\ 0 \leq t \leq \pi\}.$$

Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

$\phi_x = y^3+1 \rightarrow \int y^3+1\, dx = \underline{xy^3} + \underline{x}$

$\phi_y = 3xy^2+1 \rightarrow \int 3xy^2+1\, dy = xy^3 + \underline{y}$ unique

$\phi = xy^3 + x + y$

$xy^3 + x + y \Big|_{(0,0)}^{(2,0)} = 2$

Green's Theorem

$$\underbrace{\oint P\,dx + Q\,dy}_{\oint \vec{F}\cdot d\vec{r}\,dt} = \iint_R (Q_x - P_y)\,dA$$

- Always ccw if cw have to add (-) sign to integral

- Area: $\frac{1}{2}\oint x\,dy - y\,dx$
- Flux: $\iint_R (f_x + g_y)\,dA$

Curl and divergence

- Curl: $\text{Curl}\,\vec{F} = \vec{\nabla} \times \vec{F}$

$$\text{Curl}\,F = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix}$$

- divergence: $\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F}$

$$\text{div}\,F = \partial_x(x) + \partial_y(y) + \partial_z(z)$$

Surface integrals

Flux of surface

- Flux: $\iint \vec{F}\cdot\vec{n}\,dS$

$dS = |r_u \times r_v|\,du\,dv$

Parameterize: $\vec{r}(u,v)$

- If $z = f(x,y)$: $dS = \sqrt{1 + z_x^2 + z_y^2}\,dx\,dy$
- Don't add extra stuff $(r,\ \rho^2\sin\phi)$

- Area: $\iint dS$

$$f_{avg} = \frac{\iint F\cdot\vec{n}\,dS}{\iint dS} \rightarrow Flux_{avg} = \frac{Flux}{Area}$$

With vectors:

$\iint \vec{F}\cdot\underbrace{(r_u \times r_v)}_{n}\,dA$ Need to check if orientation is correct: plug in bounds to see if direction is correct

$n = \nabla S$

↳ surface shape bounded by

Stokes Theorem

Flux of curl of vector field

- $\iint \text{curl}\,\vec{F}\cdot\vec{n}\,dS = \oint \vec{F}\cdot dr$

Divergence Theorem

$$\underbrace{\iint \vec{F}\cdot\vec{n}\,dS}_{\text{Surface}} = \iiint_D \text{div}\,\vec{F}\,dV$$

} Divergence times volume of object

Surface: depending on how many surfaces object has, might need to do multiple and add

- If surface inside another
  - Normal vectors point in opposite directions (out and in)
  - Subtract interior integral from exterior

Line Integrals

Scalar: $\int F\,ds = \int F|r'(t)|$

Tangent: $\int F \cdot T\,ds = \int F \cdot r'(t)$ or $\oint f\,dx + g\,dy = \iint g_x - f_y\,dA$

- Work, circulation, flow, curl
- Closed loop
- ccw +, cw −

Vector: $\oint f\,dy - g\,dx = \iint f_x + g_y\,dA$

- Flux, normal, divergence
- Must be closed
- ccw +

Surface integrals

Scalar: $\iint F\,dS$

$dS = |r_u \times r_v|$

$dS = \sqrt{1 + z_x^2 + z_y^2}$

- Surface integral or surface area

Vector: $\iint F \cdot n\,dS$

$n = r_u \times r_v$ (orientation matters)

$n = \nabla S$

dS: Area

- Stoke's Theorem: $\iint \text{curl}\,F \cdot n\,dS = \oint F \cdot dr$
  - Flux of the curl
  - If $\text{curl}\,F = 0$ $\oint F \cdot dr = 0$
- Divergence Theorem: $\iint F \cdot \vec{n}\,dS = \iiint \text{div}\,F\,dV$
  - Flux
  - If $\text{div}\,F = 0$ stop