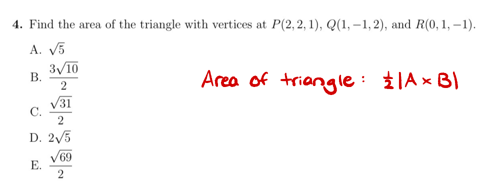
13.1 - 13.4 Review of Vectors

- S18E1 #4



4. Find the area of the triangle with vertices at $P(2,2,1)$, $Q(1,-1,2)$, and $R(0,1,-1)$.

A. $\sqrt{5}$
B. $\frac{3\sqrt{10}}{2}$
C. $\frac{\sqrt{31}}{2}$
D. $2\sqrt{5}$
E. $\frac{\sqrt{69}}{2}$

Area of triangle : $\frac{1}{2}|A \times B|$

$PQ: \langle -1, -3, 1 \rangle$

$PR: \langle -2, -1, -2 \rangle$

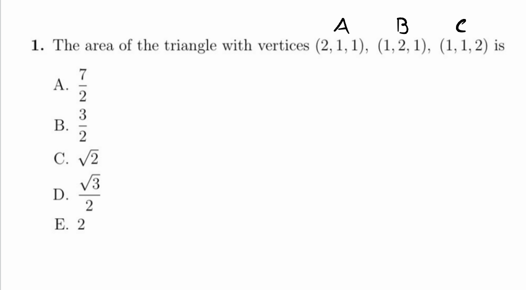
$PQ \times PR = \begin{vmatrix} -1 & -3 & 1 \\ -2 & -1 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ -1 & -2 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ -2 & -2 \end{vmatrix} + \begin{vmatrix} -1 & -3 \\ -2 & -1 \end{vmatrix}$

$\langle 6+1, -(2+2), 1-6 \rangle = \langle 7, -4, -5 \rangle$

$|PQ \times PR| = \sqrt{7^2 + (-4)^2 + (-5)^2} = \sqrt{49+16+25} = \sqrt{90} = 3\sqrt{10}$

$A = \frac{1}{2}|PQ \times PR| = \frac{3\sqrt{10}}{2}$

- S18FE #1



A B C

1. The area of the triangle with vertices $(2,1,1)$, $(1,2,1)$, $(1,1,2)$ is

A. $\frac{7}{2}$
B. $\frac{3}{2}$
C. $\sqrt{2}$
D. $\frac{\sqrt{3}}{2}$
E. 2

$AB = \langle -1, 1, 0 \rangle$

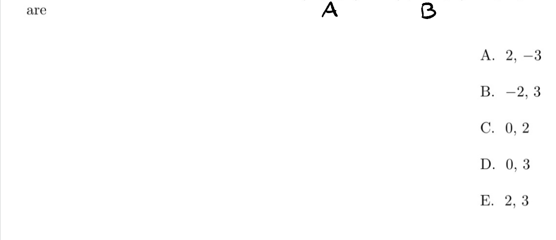
$AC = \langle -1, 0, 1 \rangle$

$AB \times AC = \begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}$

$= \langle 1, 1, 1 \rangle$

$\frac{1}{2}|AB \times AC| = \frac{1}{2}\sqrt{1^2+1^2+1^2} = \sqrt{3}/2$

- S16E1 #1



1. The two values of x for which the vectors $\langle x^2, 1, 3 \rangle$ and $\langle 1, -5x, 2 \rangle$ are perpendicular are

A B

A. 2, −3
B. −2, 3
C. 0, 2
D. 0, 3
E. 2, 3

$A \cdot B = |A||B|\cos\theta$

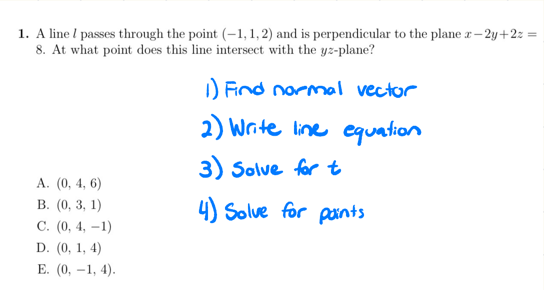
$A \cdot B = 0 \longrightarrow x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$x = 3 \quad x = 2$

13.5 Lines and Planes in Space

- S19E1#1



1. A line l passes through the point $(-1, 1, 2)$ and is perpendicular to the plane $x-2y+2z = 8$. At what point does this line intersect with the yz-plane?

A. (0, 4, 6)
B. (0, 3, 1)
C. (0, 4, −1)
D. (0, 1, 4)
E. (0, −1, 4).

1) Find normal vector
2) Write line equation
3) Solve for t
4) Solve for points

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

1) $\vec{n} = \langle 1, -2, 2 \rangle$

$r(t) = r_0 + t\vec{v}$

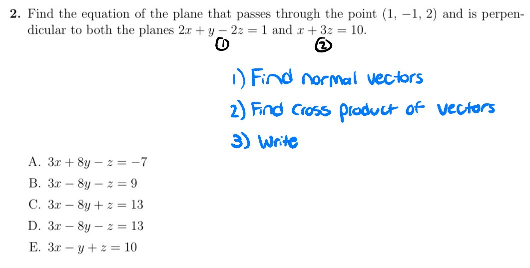
2) $= \langle -1, 1, 2 \rangle + t\langle 1, -2, 2 \rangle$

$x = t-1 \quad y = 1-2t \quad z = 2+2t$

3) $0 = t-1$

4) $t = 1 \rightarrow y = -1 \rightarrow z = 4$ $(0, -1, 4)$

- S19E1#2



2. Find the equation of the plane that passes through the point $(1, -1, 2)$ and is perpendicular to both the planes $2x + y - 2z = 1$ ① and $x + 3z = 10$. ②

A. $3x + 8y - z = -7$
B. $3x - 8y - z = 9$
C. $3x - 8y + z = 13$
D. $3x - 8y - z = 13$
E. $3x - y + z = 10$

1) Find normal vectors
2) Find cross product of vectors
3) Write

$\vec{n}_1 = \langle 2, 1, -2 \rangle$

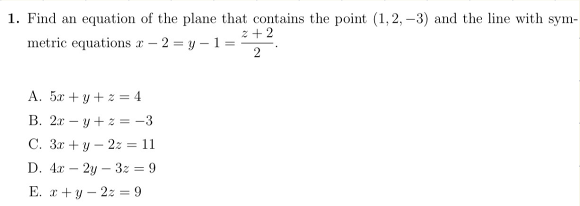
$\vec{n}_2 = \langle 1, 3, 0 \rangle$

$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} 2 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} = \langle 3, -8, -1 \rangle$

$3(x-1) - 8(y+1) - 1(z-2) =$

$3x - 8y - z = 9$

- S19FE#1



1. Find an equation of the plane that contains the point $(1, 2, -3)$ and the line with symmetric equations $x - 2 = y - 1 = \frac{z+2}{2}$.

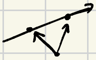
A. $5x + y + z = 4$
B. $2x - y + z = -3$
C. $3x + y - 2z = 11$
D. $4x - 2y - 3z = 9$
E. $x + y - 2z = 9$

Process

1) Find two points on line
2) Find position vector from point to line points
3) Take cross product to get vector that defines plane all 3 points are in

1) If $x = 0 : \langle 0, -1, -6 \rangle$
$x = 1 : \langle 1, 0, -4 \rangle$ } Two points on line

2) Vector from given points to line points:



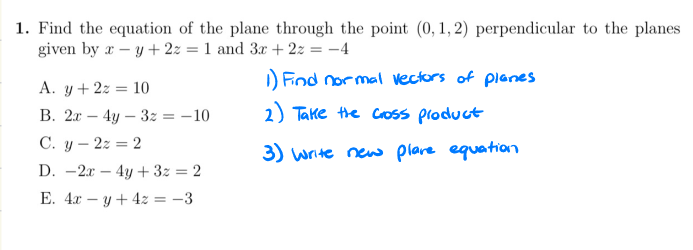
$\langle -1, -3, -3 \rangle$

$\langle 0, -2, -1 \rangle$

Cross product: $\langle -3, -1, 2 \rangle$

3) Plane : $-3(x+1) - (y-2) + 2(z+3) = 0 \longrightarrow 3x + y - 2z = 11$

- F19E1#1



1. Find the equation of the plane through the point $(0, 1, 2)$ perpendicular to the planes given by $x - y + 2z = 1$ and $3x + 2z = -4$

A. $y + 2z = 10$

B. $2x - 4y - 3z = -10$

C. $y - 2z = 2$

D. $-2x - 4y + 3z = 2$

E. $4x - y + 4z = -3$

1) Find normal vectors of planes
2) Take the cross product
3) Write new plane equation

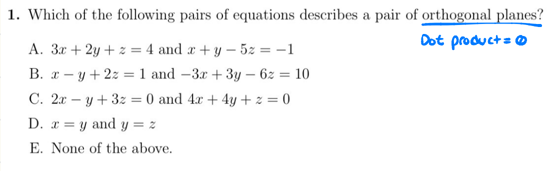
$A: \langle 1, -1, 2 \rangle \qquad B: \langle 3, 0, 2 \rangle$

$A \times B = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \end{vmatrix} = \langle -2, 4, 3 \rangle$

$-2x + 4(y-1) + 3(z-2) = 0$

$-2x + 4y + 3z = 10 \longrightarrow 2x - 4y - 3z = -10$

- F19FE#1



1. Which of the following pairs of equations describes a pair of orthogonal planes?

Dot product = 0

A. $3x + 2y + z = 4$ and $x + y - 5z = -1$

B. $x - y + 2z = 1$ and $-3x + 3y - 6z = 10$

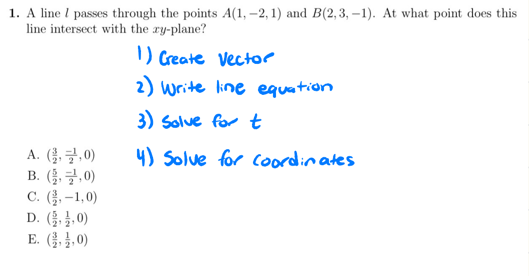
C. $2x - y + 3z = 0$ and $4x + 4y + z = 0$

D. $x = y$ and $y = z$

E. None of the above.

A) $\langle 3, 2, 1 \rangle \cdot \langle 1, 1, -5 \rangle = 0$

- F18E1#1



1. A line l passes through the points $A(1, -2, 1)$ and $B(2, 3, -1)$. At what point does this line intersect with the xy-plane?

1) Create vector
2) Write line equation
3) Solve for t
4) Solve for coordinates

A. $(\frac{3}{2}, \frac{-1}{2}, 0)$

B. $(\frac{5}{2}, \frac{-1}{2}, 0)$

C. $(\frac{3}{2}, -1, 0)$

D. $(\frac{5}{2}, \frac{1}{2}, 0)$

E. $(\frac{3}{2}, \frac{1}{2}, 0)$

$r(t) = r_0 + t\vec{v}$

$AB = \langle 1, 5, -2 \rangle$

$r(t) = \langle 1, -2, 1 \rangle + t \langle 1, 5, -2 \rangle$

$x = 1 + t$

$y = -2 + 5t$

$z = 1 - 2t = 0$

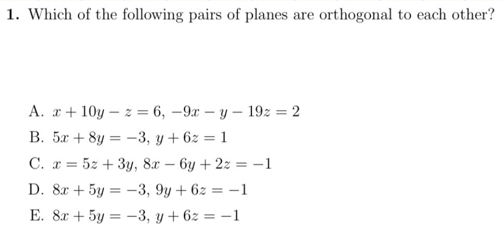
$t = 1/2$

$x = 3/2$

$y = -\frac{4}{2} + \frac{5}{2} = \frac{1}{2}$

$(3/2, 1/2, 0)$

- F18FE#1



1. Which of the following pairs of planes are orthogonal to each other?

A. $x + 10y - z = 6, \ -9x - y - 19z = 2$

B. $5x + 8y = -3, \ y + 6z = 1$

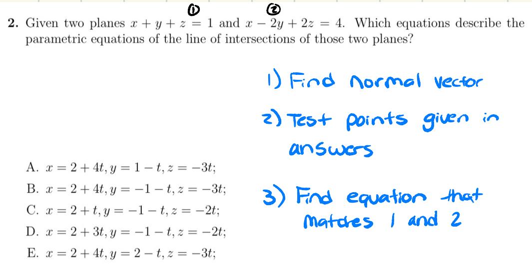
C. $x = 5z + 3y, \ 8x - 6y + 2z = -1$

D. $8x + 5y = -3, \ 9y + 6z = -1$

E. $8x + 5y = -3, \ y + 6z = -1$

A) $\langle 1, 10, -1 \rangle \cdot \langle -9, -1, -19 \rangle = -9 - 10 + 19 = 0$

· F18E1#2



2. Given two planes $\overset{①}{x + y + z = 1}$ and $\overset{②}{x - 2y + 2z = 4}$. Which equations describe the parametric equations of the line of intersections of those two planes?

A. $x = 2 + 4t, y = 1 - t, z = -3t$;
B. $x = 2 + 4t, y = -1 - t, z = -3t$;
C. $x = 2 + t, y = -1 - t, z = -2t$;
D. $x = 2 + 3t, y = -1 - t, z = -2t$;
E. $x = 2 + 4t, y = 2 - t, z = -3t$;

1) Find normal vector

2) Test points given in answers

3) Find equation that matches 1 and 2

①: $\langle 1, 1, 1\rangle$ ②: $\langle 1, -2, 2\rangle$

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \end{vmatrix} = \langle 4, -1, -3\rangle$

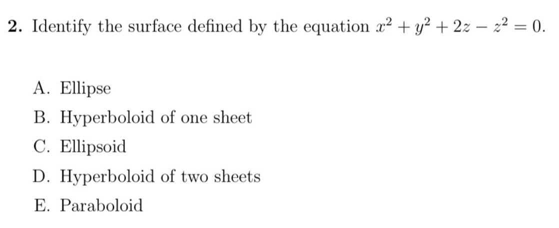
A) P: (2, 1, -3) ①: ✓ ②: X

C) P: (2, -1, 0) ①: ✓ ②: ✓

B

13.6 Quadratic Surfaces

· S19FE #2



2. Identify the surface defined by the equation $x^2 + y^2 + 2z - z^2 = 0$.

A. Ellipse
B. Hyperboloid of one sheet
C. Ellipsoid
D. Hyperboloid of two sheets
E. Paraboloid

$x^2 + y^2 - z^2 + 2z = 0$

$x^2 + y^2 - (z^2 - 2z) = 0$

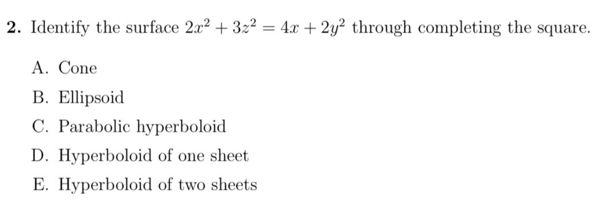
$x^2 + y^2 - (z^2 - 2z - 1) = -1$

$-x^2 - y^2 + (z-1)^2 = 1$

Hyperboloid of two sheets

Be aware of signs

· F19E1 #2



2. Identify the surface $2x^2 + 3z^2 = 4x + 2y^2$ through completing the square.

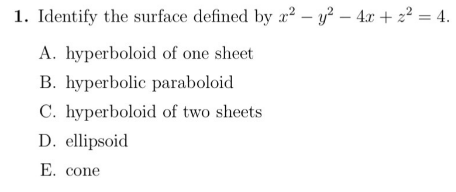
A. Cone
B. Ellipsoid
C. Parabolic hyperboloid
D. Hyperboloid of one sheet
E. Hyperboloid of two sheets

$2x^2 - 4x + \underline{4} - 2y^2 + 3z = 0$

$\frac{(2x-2)^2}{4} - \frac{2y^2}{4} + \frac{3z}{4} = 0$

Hyperboloid of one sheet

· S18E1#1



1. Identify the surface defined by $x^2 - y^2 - 4x + z^2 = 4$.

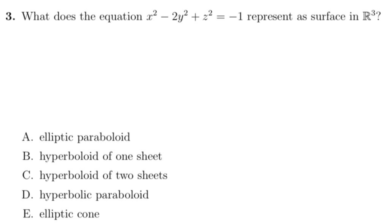
A. hyperboloid of one sheet
B. hyperbolic paraboloid
C. hyperboloid of two sheets
D. ellipsoid
E. cone

$x^2 - 4x + \_\_ - y^2 + z^2 = 4$

$\frac{(x-2)^2}{8} - \frac{y^2}{8} + \frac{z^2}{8} = 1$

Hyperboloid of one sheet

- F18E1#3



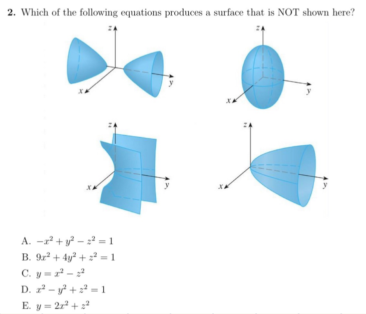
3. What does the equation $x^2 - 2y^2 + z^2 = -1$ represent as surface in $\mathbb{R}^3$?

A. elliptic paraboloid
B. hyperboloid of one sheet
C. hyperboloid of two sheets
D. hyperbolic paraboloid
E. elliptic cone

$-x^2 + 2y^2 - z^2 = 1$

Hyperboloid of two sheets

- F18FE#2



2. Which of the following equations produces a surface that is NOT shown here?

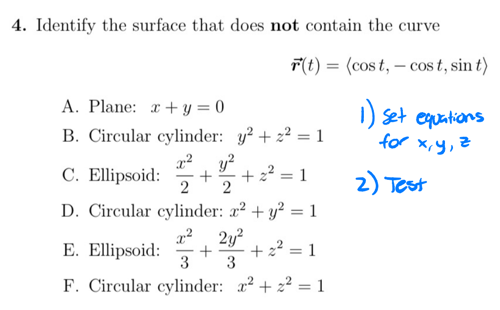
A. $-x^2 + y^2 - z^2 = 1$
B. $9x^2 + 4y^2 + z^2 = 1$
C. $y = x^2 - z^2$
D. $x^2 - y^2 + z^2 = 1$
E. $y = 2x^2 + z^2$

A) Hyper. of two sheets
B) Ellipsoid
C) Hyperbolic parab
[✓ D)] Hyper. one sheet
E) parabo

14.1 Vector-Valued Functions

Needs Practice

- S22E1#4



4. Identify the surface that does **not** contain the curve

$$\vec{r}(t) = \langle \cos t, -\cos t, \sin t \rangle$$

A. Plane: $x + y = 0$
B. Circular cylinder: $y^2 + z^2 = 1$
C. Ellipsoid: $\frac{x^2}{2} + \frac{y^2}{2} + z^2 = 1$
D. Circular cylinder: $x^2 + y^2 = 1$
E. Ellipsoid: $\frac{x^2}{3} + \frac{2y^2}{3} + z^2 = 1$
F. Circular cylinder: $x^2 + z^2 = 1$

1) set equations for x, y, z

2) Test

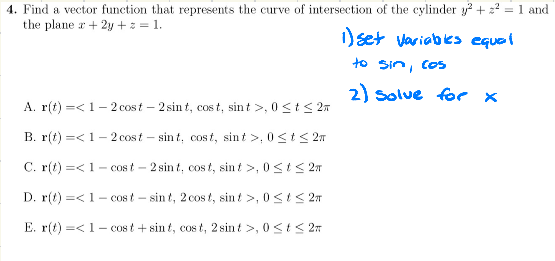
A) $\cos t - \cos t = 0$ ✓

B) $\cos^2 t + \sin^2 t = 1$ ✓

C) $\frac{\cos^2 t}{2} + \frac{\cos^2 t}{2} + \sin^2 t = 1$ ✓

[✓ D)] $\cos^2 t + \cos^2 t \neq 1$

- S19E1#3



4. Find a vector function that represents the curve of intersection of the cylinder $y^2 + z^2 = 1$ and the plane $x + 2y + z = 1$.

A. $\mathbf{r}(t) =< 1 - 2\cos t - 2\sin t,\ \cos t,\ \sin t >,\ 0 \le t \le 2\pi$
B. $\mathbf{r}(t) =< 1 - 2\cos t - \sin t,\ \cos t,\ \sin t >,\ 0 \le t \le 2\pi$
C. $\mathbf{r}(t) =< 1 - \cos t - 2\sin t,\ \cos t,\ \sin t >,\ 0 \le t \le 2\pi$
D. $\mathbf{r}(t) =< 1 - \cos t - \sin t,\ 2\cos t,\ \sin t >,\ 0 \le t \le 2\pi$
E. $\mathbf{r}(t) =< 1 - \cos t + \sin t,\ \cos t,\ 2\sin t >,\ 0 \le t \le 2\pi$

1) set variables equal to sin, cos

2) solve for x

$y^2 + z^2 = 1$

$\sin^2 + \cos^2 = 1$

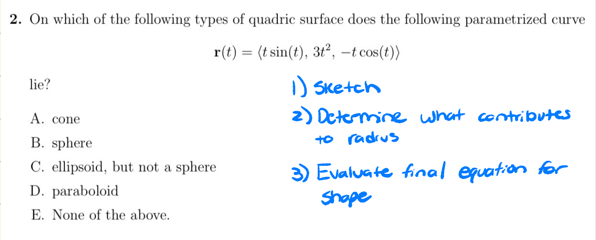
$x = -2\sin t - \cos t + 1$

or

$x = 1 - 2\cos t - \sin t$

[✓ B]

· F19FE #2



2. On which of the following types of quadric surface does the following parametrized curve

$$\mathbf{r}(t) = \langle t\sin(t),\ 3t^2,\ -t\cos(t)\rangle$$

lie?

A. cone

B. sphere

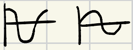
C. ellipsoid, but not a sphere

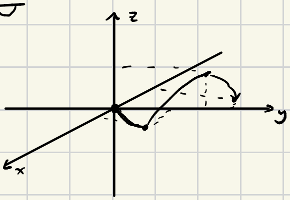
D. paraboloid

E. None of the above.

1) Sketch
2) Determine what contributes to radius
3) Evaluate final equation for shape

t	x	y	z
0	0	0	0
$\pi/2$	$\pi/2$	$\frac{3\pi^2}{4}$	0
π	0	$3\pi^2$	π
$3\pi/2$	$-\frac{3\pi}{2}$	$\frac{27\pi^2}{4}$	0
2π	0	$12\pi^2$	-2π





Spiral : Cone or paraboloid

To determine, test Quadric surfice equations:

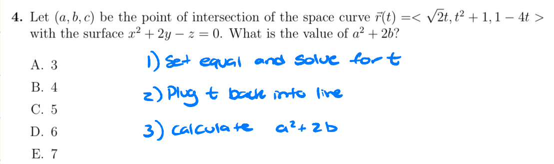
Cone : $x^2+y^2=z^2$

paraboloid : $\frac{x^2}{a}+\frac{y^2}{b}=z$

$$\left.\begin{aligned} x &= t\sin(t) \\ z &= -t\cos(t) \end{aligned}\right\} \text{Radius}$$

$y = 3t^2 \longrightarrow$ Parabola

· F16 E1 #4



4. Let (a, b, c) be the point of intersection of the space curve $\vec{r}(t) =< \sqrt{2}t, t^2+1, 1-4t >$ with the surface $x^2 + 2y - z = 0$. What is the value of $a^2 + 2b$?

A. 3

B. 4

C. 5

D. 6

E. 7

1) Set equal and solve for t
2) Plug t back into line
3) calculate a^2+2b

$$2t^2+2t^2+2-1+4t=0 \longrightarrow 4t^2+4t+1=0$$

$$(4t^2+2t)(2t+1)=0$$

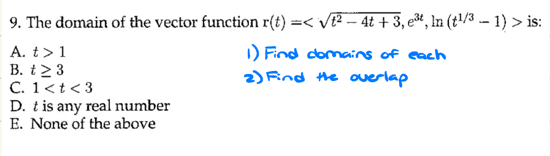
$$2t(2t+1)+1(2t+1)=0$$

$$(2t+1)^2=0 \qquad t=-1/2$$

$$r(t) = \langle -\sqrt{2}/2,\ 5/4,\ 3 \rangle$$

$$a^2+b = \frac{2}{4}+\frac{10}{4} = 3$$

• S14E1 #9



9. The domain of the vector function $\mathbf{r}(t) =< \sqrt{t^2 - 4t + 3}, e^{3t}, \ln(t^{1/3} - 1) >$ is:

A. $t > 1$
B. $t \geq 3$
C. $1 < t < 3$
D. t is any real number
E. None of the above

1) Find domains of each
2) Find the overlap

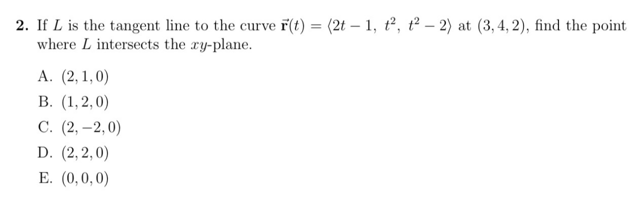
$x = \sqrt{t^2-4t+3}$ $(t-3)(t-1)$ $t > 0$

$y = e^{3t}$ $-\infty < t < \infty$ $t \geq 3$

$z = \ln(t^{1/3}-1)$ $t \geq 3$

14.2-3 Calculus of Vector-Valued Functions & Motion in space

• S18E1 #2



2. If L is the tangent line to the curve $\vec{r}(t) = \langle 2t-1,\ t^2,\ t^2-2\rangle$ at $(3,4,2)$, find the point where L intersects the xy-plane.

A. $(2,1,0)$
B. $(1,2,0)$
C. $(2,-2,0)$
D. $(2,2,0)$
E. $(0,0,0)$

Think of $\vec{v}$ as the slope

$r(t) = \vec{r}_0 + \vec{v}t$
$= \vec{r}_0 + r'(t)t$

$\vec{r}_0 = \langle 3,4,2\rangle$ 1) $2t-1=3$
$t=2$

1) Solve for t
2) Solve for the derivative and plug in t to get $\vec{v}$
3) Write new equation using $\vec{v}$ and given point
4) Solve for points

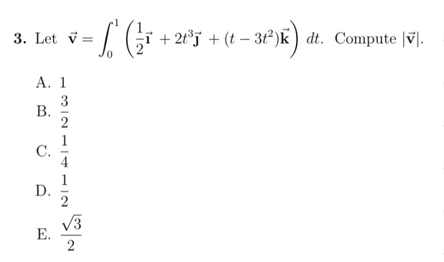
2) $r'(t) = \langle 2, 2t, 2t\rangle =$
$r'(2) = \langle 2,4,4\rangle = \vec{v}$

3) $L(t) = \langle 3,4,2\rangle + t\langle 2,4,4\rangle = \langle 2t+3, 4t+4, 4t+2\rangle$

$4t+2=0$ $x = 2(-1/2)+3$ $y = 4(-1/2)+4$
$t = -1/2$ $x = 2$ $y = 2$

4) $(2,2,0)$

• S18E1 #3

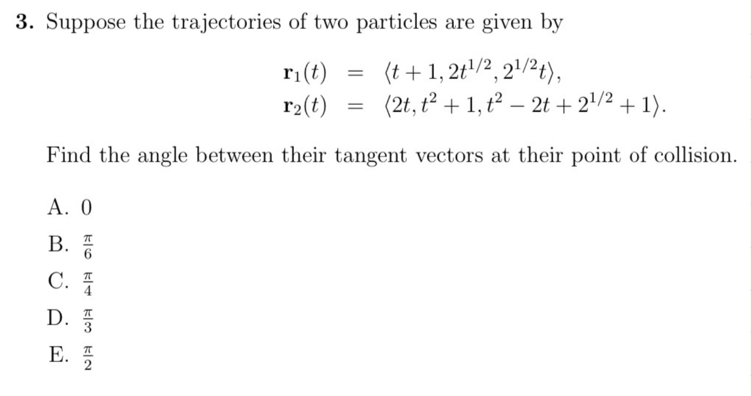


3. Let $\vec{\mathbf{v}} = \int_0^1 \left(\frac{1}{2}\vec{\imath} + 2t^3\vec{\jmath} + (t-3t^2)\vec{\mathbf{k}}\right) dt$. Compute $|\vec{\mathbf{v}}|$.

A. 1
B. $\frac{3}{2}$
C. $\frac{1}{4}$
D. $\frac{1}{2}$
E. $\frac{\sqrt{3}}{2}$

$\vec{v} = \langle \frac{1}{2}t\Big|_0^1, \frac{2}{4}t^4\Big|_0^1, \frac{1}{2}t^2 - t^3\Big|_0^1 \rangle$
$= \langle \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle$
$|\vec{v}| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

· S17E1 #3



3. Suppose the trajectories of two particles are given by

$$\mathbf{r}_1(t) = \langle t+1, 2t^{1/2}, 2^{1/2}t\rangle,$$
$$\mathbf{r}_2(t) = \langle 2t, t^2+1, t^2-2t+2^{1/2}+1\rangle.$$

Find the angle between their tangent vectors at their point of collision.

A. 0

B. $\frac{\pi}{6}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{3}$

E. $\frac{\pi}{2}$

$r_1'(t) = \langle 1, \frac{1}{t^{1/2}}, 2^{1/2} \rangle$

$r_2'(t) = \langle 2, 2t, 2t-2 \rangle$

$r_1(t) = r_2(t) \longrightarrow 2t = t+1 \longrightarrow t = 1$

$r_1'(1) = \langle 1, 1, \sqrt{2} \rangle$

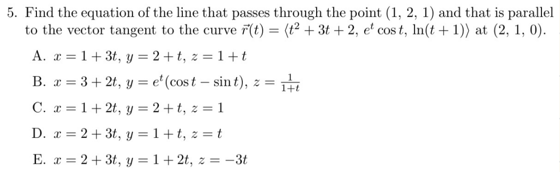
$r_2'(1) = \langle 2, 2, 0 \rangle$

$\langle 1, 1, \sqrt{2} \rangle \cdot \langle 2, 2, 0 \rangle = \sqrt{1+1+2} \cdot \sqrt{4+4} \cos\theta$

$4 = 4\sqrt{2} \cos\theta$

$\frac{1}{\sqrt{2}} = \cos\theta \longrightarrow \theta = \pi/4$

· S16E1 #5



5. Find the equation of the line that passes through the point (1, 2, 1) and that is parallel to the vector tangent to the curve $\vec{r}(t) = \langle t^2+3t+2, e^t\cos t, \ln(t+1)\rangle$ at (2, 1, 0).

A. $x = 1+3t$, $y = 2+t$, $z = 1+t$

B. $x = 3+2t$, $y = e^t(\cos t - \sin t)$, $z = \frac{1}{1+t}$

C. $x = 1+2t$, $y = 2+t$, $z = 1$

D. $x = 2+3t$, $y = 1+t$, $z = t$

E. $x = 2+3t$, $y = 1+2t$, $z = -3t$

$r'(t) = \langle 2t+3, -\sin t e^t + e^t \cos t, \frac{1}{t+1} \rangle$

$t^2+3t+2 = 2 \qquad e^0 \cos(0) = 1$ ✓

$t(t+3) = 0$

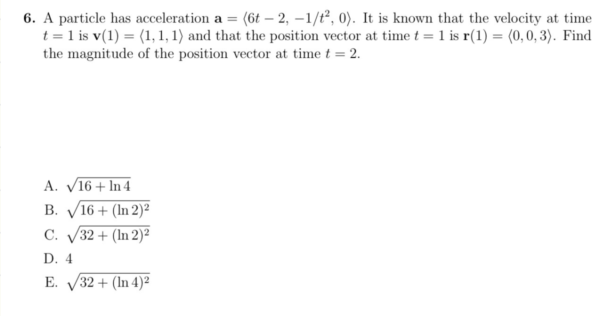
$t = 0 \quad t = -3$

$r'(0) = \vec{v} = \langle 3, 1, 1 \rangle$

$\vec{L}(t) = \langle 1, 2, 1 \rangle + t\langle 3, 1, 1 \rangle = \langle 3t+1, t+2, t+1 \rangle$

14.3 Motion in Space

- S19 E1 #6



6. A particle has acceleration $\mathbf{a} = \langle 6t - 2, -1/t^2, 0\rangle$. It is known that the velocity at time $t = 1$ is $\mathbf{v}(1) = \langle 1, 1, 1\rangle$ and that the position vector at time $t = 1$ is $\mathbf{r}(1) = \langle 0, 0, 3\rangle$. Find the magnitude of the position vector at time $t = 2$.

A. $\sqrt{16 + \ln 4}$
B. $\sqrt{16 + (\ln 2)^2}$
C. $\sqrt{32 + (\ln 2)^2}$
D. 4
E. $\sqrt{32 + (\ln 4)^2}$

$v(t) = \int a(t) = \langle 3t^2 - 2t, \frac{1}{t}, 0\rangle + \langle c, c, c\rangle$

$v(1)_x = 3 - 2 = 1 \longrightarrow C = 0$

$v(1)_y = 1 + C = 1 \longrightarrow C = 0$

$v(1)_z = 0 + C = 1 \longrightarrow C = 1$

$v(t) = \langle 3t^2 - 2t, \frac{1}{t}, 1\rangle$

$r'(t) = \langle t^3 - t^2, \ln(t), t\rangle + \langle c, c, c\rangle$

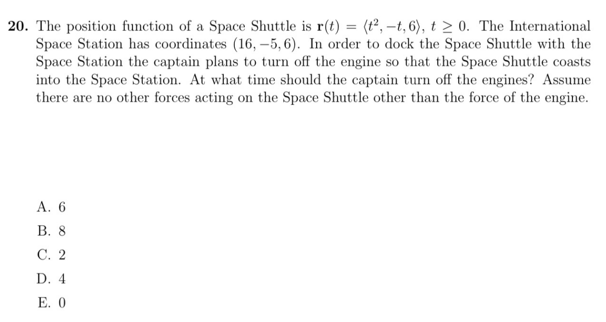
$C_x = 0 \quad C_y = 0 \quad C_z = 2$

$r(t) = \langle t^3 - t^2, \ln(t), t + 2\rangle$

$r(2) = \langle 4, \ln(2), 4\rangle$

$|r(2)| = \sqrt{32 + [\ln(2)]^2}$

- S19 FE #20



20. The position function of a Space Shuttle is $\mathbf{r}(t) = \langle t^2, -t, 6\rangle$, $t \geq 0$. The International Space Station has coordinates $(16, -5, 6)$. In order to dock the Space Shuttle with the Space Station the captain plans to turn off the engine so that the Space Shuttle coasts into the Space Station. At what time should the captain turn off the engines? Assume there are no other forces acting on the Space Shuttle other than the force of the engine.

A. 6
B. 8
C. 2
D. 4
E. 0

1) Find Velocity and acceleration
2) Determine time it will take to get from Current position to desired (go with simplest t)
3) Rule out what is greater than t than test rest with velocity and position

Note: velocity is how many units you move per second

$r(t) = \langle t^2, -t, 6\rangle \quad t \geq 0$

$v(t) = \langle 2t, -1, 0\rangle$

$a(t) = \langle 2, 0, 0\rangle$

At $t = 5$ ship will coast past station on y-axis

$r(5) = \langle 25, -5, 6\rangle \quad \therefore t \neq 5$

b/c will coast past in x-direction

If $t = 0$ $a = \langle 0, 0, 0\rangle$

won't move at all

If $t = 4$ $v(4) = \langle 8, -1, 0\rangle$

$r(4) = \langle 16, -4, 6\rangle$

Not good b/c over next second will move 8 units on x-axis

If $t = 2$ $v(2) = \langle 4, -1, 0\rangle$

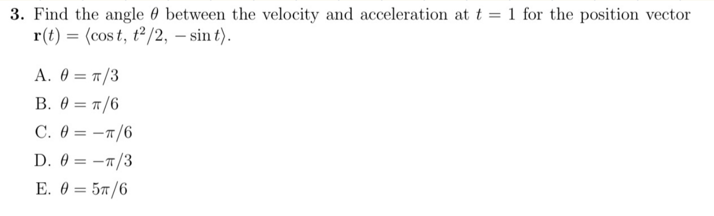
$r(2) = \langle 4, -2, 6\rangle$

3 seconds remaining

$3 \cdot v(2) = \langle 12, -3, 0\rangle = a$

$a \cdot r(2) = \langle 16, -5, 6\rangle$

· F19E1#3



3. Find the angle θ between the velocity and acceleration at $t = 1$ for the position vector $\mathbf{r}(t) = \langle \cos t,\, t^2/2,\, -\sin t \rangle$.

A. $\theta = \pi/3$

B. $\theta = \pi/6$

C. $\theta = -\pi/6$

D. $\theta = -\pi/3$

E. $\theta = 5\pi/6$

$v(t) = \langle -\sin t,\ t,\ -\cos t \rangle$ $\quad$ $V(1) = \langle -\sin(1),\ 1,\ -\cos(1) \rangle$

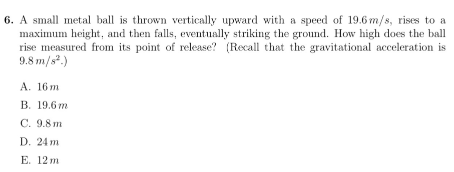
$a(t) = \langle -\cos t,\ 1,\ \sin t \rangle$ $\quad$ $a(1) = \langle -\cos(1),\ 1,\ \sin(1) \rangle$

$V \cdot a = |v|\,|a| \cos\theta$

$\cos(1)\sin(1) + 1 - \sin(1)\cos(1) = \sqrt{2}\,\sqrt{2}\,\cos\theta$

$\frac{1}{2} = \cos\theta \longrightarrow \theta = \pi/3$

· F19E1#6



6. A small metal ball is thrown vertically upward with a speed of $19.6\,m/s$, rises to a maximum height, and then falls, eventually striking the ground. How high does the ball rise measured from its point of release? (Recall that the gravitational acceleration is $9.8\,m/s^2$.)

A. $16\,m$

B. $19.6\,m$

C. $9.8\,m$

D. $24\,m$

E. $12\,m$

1) Integrate to find velocity vector, set $t=0$ solve for c

2) Solve for r

3) Set $V=0$ solve for t

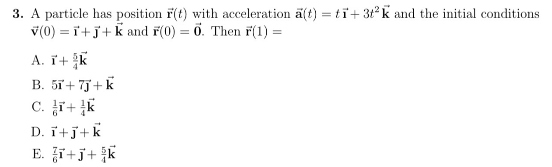
$a = \langle 0, 0, -9.8 \rangle$

$V = \langle 0, 0, -9.8t \rangle + \langle 0, 0, c \rangle = 19.6$

$V = \langle 0, 0, -9.8t + 19.6 \rangle \longrightarrow t = 2$

$r = \langle 0, 0, -4.9t^2 + 19.6t \rangle \longrightarrow r(2) = 19.6$

· S18FE #3



3. A particle has position $\vec{\mathbf{r}}(t)$ with acceleration $\vec{\mathbf{a}}(t) = t\,\vec{\mathbf{\imath}} + 3t^2\,\vec{\mathbf{k}}$ and the initial conditions $\vec{\mathbf{v}}(0) = \vec{\mathbf{\imath}} + \vec{\mathbf{\jmath}} + \vec{\mathbf{k}}$ and $\vec{\mathbf{r}}(0) = \vec{\mathbf{0}}$. Then $\vec{\mathbf{r}}(1) =$

A. $\vec{\mathbf{\imath}} + \frac{5}{4}\vec{\mathbf{k}}$

B. $5\vec{\mathbf{\imath}} + 7\vec{\mathbf{\jmath}} + \vec{\mathbf{k}}$

C. $\frac{1}{6}\vec{\mathbf{\imath}} + \frac{1}{4}\vec{\mathbf{k}}$

D. $\vec{\mathbf{\imath}} + \vec{\mathbf{\jmath}} + \vec{\mathbf{k}}$

E. $\frac{7}{6}\vec{\mathbf{\imath}} + \vec{\mathbf{\jmath}} + \frac{5}{4}\vec{\mathbf{k}}$

$a = \langle t, 0, 3t^2 \rangle$ $\quad$ $V(0) = \langle 1, 1, 1 \rangle$ $\quad$ $r(0) = \langle 0, 0, 0 \rangle$

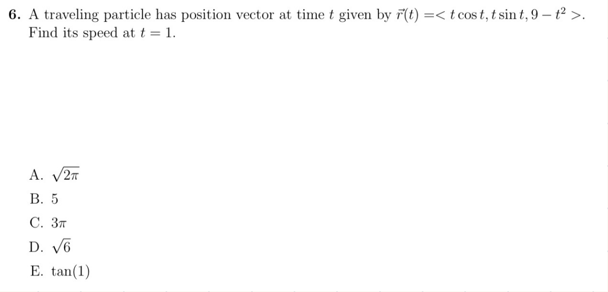
$V = \langle \frac{1}{2}t^2, 0, t^3 \rangle + \langle c, c, c \rangle$

$= \langle \frac{1}{2}t^2 + 1,\ 1,\ t^3 + 1 \rangle$

$r(t) = \langle \frac{1}{6}t^3 + t,\ t,\ \frac{1}{4}t^4 + t \rangle$

$r(1) = \langle 7/6,\ 1,\ 5/4 \rangle$

- F18E1 #6



6. A traveling particle has position vector at time t given by $\vec{r}(t) = < t\cos t, t\sin t, 9 - t^2 >$. Find its speed at $t = 1$.

A. $\sqrt{2\pi}$
B. 5
C. 3π
D. $\sqrt{6}$
E. $\tan(1)$

$$v(t) = \langle -t\sin t + \cos t,\ t\cos t + \sin t,\ -2t \rangle$$

$$v(1) = \langle -\sin(1) + \cos(1),\ \cos(1) + \sin(1),\ -2 \rangle$$

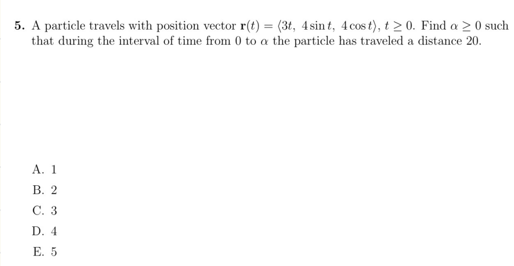
$$|v(1)| = \sqrt{[\sin^2(1) - 2\sin(1)\cos(1) + \cos^2(1)] + [\cos^2(1) + 2\cos(1)\sin(1) + \sin^2(1)] + 4}$$

$$= \sqrt{1+1+4} = \sqrt{6}$$

14.4-14.5 Length of Curves / Curvature

Arc length $L = \int_0^b |r'(t)|\,dt$

- S19E1 #5



5. A particle travels with position vector $\mathbf{r}(t) = \langle 3t,\ 4\sin t,\ 4\cos t \rangle$, $t \geq 0$. Find $\alpha \geq 0$ such that during the interval of time from 0 to α the particle has traveled a distance 20.

A. 1
B. 2
C. 3
D. 4
E. 5

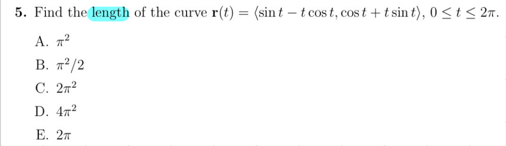
$$r'(t) = \langle 3, 4\cos t, -4\sin t \rangle$$

$$|r'(t)| = \sqrt{9 + 16} = 5$$

$$L = 20 = \int_0^\alpha 5\,dt = 5\alpha = 20$$

$$\alpha = 4$$

- F19E1 #5



5. Find the length of the curve $\mathbf{r}(t) = \langle \sin t - t\cos t, \cos t + t\sin t \rangle$, $0 \leq t \leq 2\pi$.

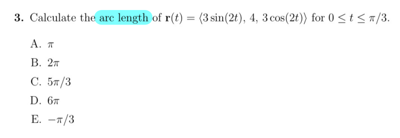
A. π^2
B. $\pi^2/2$
C. $2\pi^2$
D. $4\pi^2$
E. 2π

$$r'(t) = \langle \cancel{\cos t} + t\sin t - \cancel{\cos t},\ \cancel{-\sin t} + t\cos t + \cancel{\sin t} \rangle$$

$|r'(t)| = \sqrt{t^2\sin^2 t + t^2\cos^2 t} = t$

$L = \int_0^{2\pi} t = \frac{1}{2}t^2 \Big|_0^{2\pi} = \frac{4\pi^2}{2} = 2\pi^2$

· F19FE#3



3. Calculate the arc length of $\mathbf{r}(t) = \langle 3\sin(2t),\ 4,\ 3\cos(2t)\rangle$ for $0 \le t \le \pi/3$.

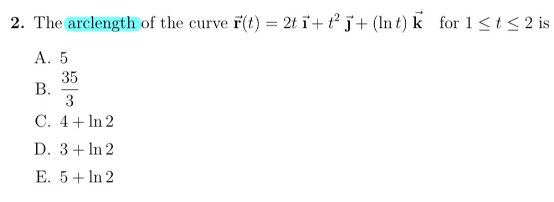
A. π
B. 2π
C. $5\pi/3$
D. 6π
E. $-\pi/3$

$r'(t) = \langle 6\sin 2t,\ 0,\ 6\cos 2t \rangle$

$|r'(t)| = \sqrt{36\sin^2 2t + 36\cos^2 2t} = 6$

$L = \int_0^{\pi/3} 6 = 2\pi$

· S18FE#2



2. The arclength of the curve $\vec{\mathbf{r}}(t) = 2t\,\vec{\mathbf{i}} + t^2\,\vec{\mathbf{j}} + (\ln t)\,\vec{\mathbf{k}}$ for $1 \le t \le 2$ is

A. 5
B. $\frac{35}{3}$
C. $4 + \ln 2$
D. $3 + \ln 2$
E. $5 + \ln 2$

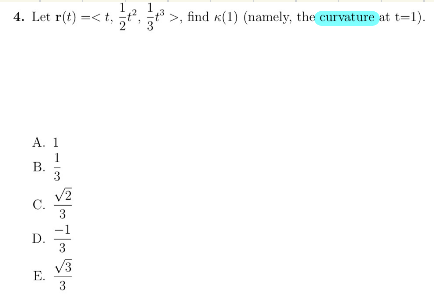
$r'(t) = \langle 2,\ 2t,\ \frac{1}{t} \rangle$

$|r'(t)| = \sqrt{4 + 4t^2 + \frac{1}{t^2}} = 2t + \frac{1}{t}$

$\int_1^2 2t + \frac{1}{t} = t^2 + \ln t \Big|_1^2 = 4 + \ln 2 - 1 = 3 + \ln(2)$

Curvature

· S19E1#4



4. Let $\mathbf{r}(t) =< t, \frac{1}{2}t^2, \frac{1}{3}t^3 >$, find $\kappa(1)$ (namely, the curvature at t=1).

A. 1
B. $\frac{1}{3}$
C. $\frac{\sqrt{2}}{3}$
D. $\frac{-1}{3}$
E. $\frac{\sqrt{3}}{3}$

$K = \frac{|T'|}{|r'|} = \frac{|r'' \times r'|}{|r'|^3}$

$r'(t) = \langle 1,\ t,\ t^2 \rangle$

$|r'(t)| = \sqrt{1 + t^2 + t^4}$ $\quad t = 1 \quad \sqrt{3}$

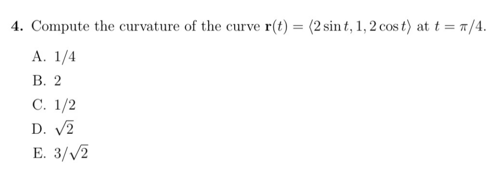
$r''(t) = \langle 0,\ 1,\ 2t \rangle$

$r'' \times r' = \begin{vmatrix} 0 & 1 & 2t \\ 1 & t & t^2 \end{vmatrix} = \langle t^2 - 2t^2,\ 2t,\ -1 \rangle = \langle -t^2,\ 2t,\ -1 \rangle = \sqrt{6}$

$|r'|^3 = 3\sqrt{3}$

$K = \frac{\sqrt{6}}{\sqrt{3}} = \frac{\sqrt{2}\cdot\sqrt{3}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}$

• F19E1 #4


4. Compute the curvature of the curve $\mathbf{r}(t) = \langle 2\sin t, 1, 2\cos t\rangle$ at $t = \pi/4$.

- A. 1/4
- B. 2
- C. 1/2
- D. $\sqrt{2}$
- E. $3/\sqrt{2}$

$r' = \langle 2\cos t, 0, -2\sin t\rangle$

$|r'| = \sqrt{4\cos^2 t + 4\sin^2 t} = 2$

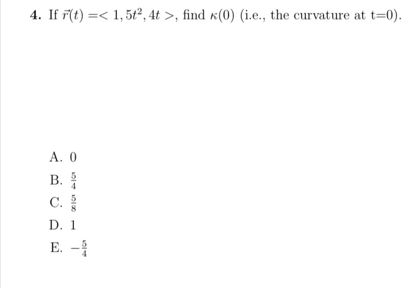
$T = \langle \cos t, 0, -\sin t\rangle$

$T' = \langle -\sin t, 0, \cos t\rangle$

$|T'| = 1$

$K = \frac{1}{2}$

• F18E1 #4


4. If $\vec{r}(t) =< 1, 5t^2, 4t >$, find $\kappa(0)$ (i.e., the curvature at t=0).

- A. 0
- B. $\frac{5}{4}$
- C. $\frac{5}{8}$
- D. 1
- E. $-\frac{5}{4}$

$r' = \langle 0, 10t, 4\rangle$

$r'' = \langle 0, 10, 0\rangle$

$r'' \times r' = \begin{vmatrix} 0 & 10 & 0 \\ 0 & 10t & 4 \end{vmatrix} = \langle 40, 0, 0\rangle$

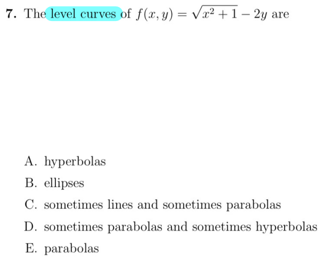
$|r'| = \sqrt{100t^2 + 16} = 4$

$|r'|^3 = 64$

$K = \frac{40}{64} = \frac{5}{8}$

15.1 Functions of Several Variables

• S19E1 #7


7. The level curves of $f(x, y) = \sqrt{x^2 + 1} - 2y$ are

- A. hyperbolas
- B. ellipses
- C. sometimes lines and sometimes parabolas
- D. sometimes parabolas and sometimes hyperbolas
- E. parabolas

$4y^2 = x^2 + 1$

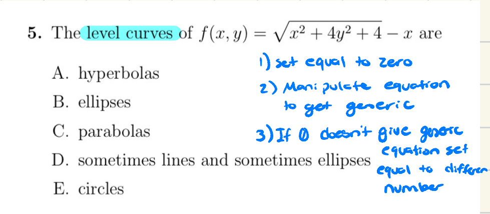
$4y^2 - x^2 = 1$

Hyperbolas

Hyperbolas: $x^2 - y^2 = c$

Parabolas: $y = x^2$ or $x = y^2$

- S18E1#5



5. The level curves of $f(x,y)=\sqrt{x^2+4y^2+4}-x$ are

A. hyperbolas
B. ellipses
C. parabolas
D. sometimes lines and sometimes ellipses
E. circles

1) set equal to zero
2) Manipulate equation to get generic
3) If 0 doesn't give generic equation set equal to different number

$x=\sqrt{x^2+4y^2+4}$

$x^2=x^2+4y^2+4 \longrightarrow -4=4y^2$

$(x+1)^2=x^2+4y^2+4$

$x^2+2x+1=x^2+4y^2+4$

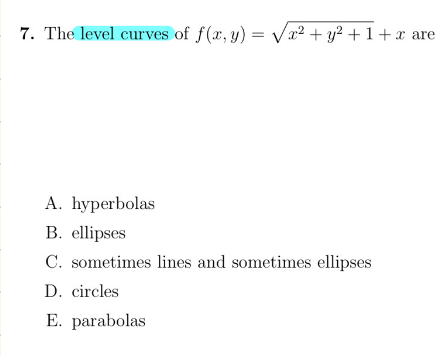
$(x+2)^2=x^2+4y^2+4$

$x^2+4x+4=x^2+4y^2+4$

$4x=4y^2 \longrightarrow x=y^2$

Parabolas

- F18E1#7



7. The level curves of $f(x,y)=\sqrt{x^2+y^2+1}+x$ are

A. hyperbolas
B. ellipses
C. sometimes lines and sometimes ellipses
D. circles
E. parabolas

$x^2=x^2+y^2+1$

$y^2=-1$

$(x+1)^2=x^2+y^2+1$

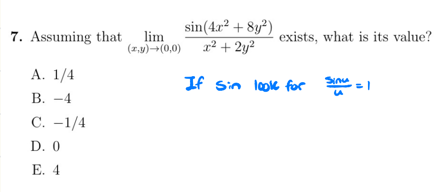
$x^2+2x+1=x^2+y^2+1$

$y^2=2x$

Parabolas

15.2 Limits and Continuity

- F19E1#7



7. Assuming that $\lim_{(x,y)\to(0,0)} \frac{\sin(4x^2+8y^2)}{x^2+2y^2}$ exists, what is its value?

A. 1/4
B. −4
C. −1/4
D. 0
E. 4

If sin look for $\frac{\sin u}{u}=1$

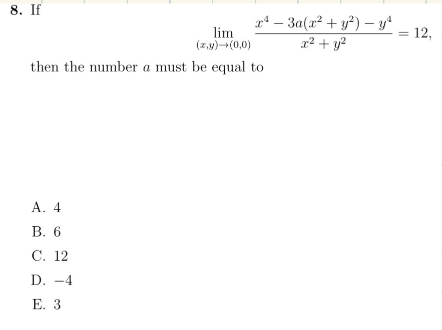
x-axis : $\frac{\sin(4x^2)}{x^2}=\frac{0}{0}$

y-axis : $\frac{\sin(8y^2)}{2y^2}=\frac{0}{0}$

$y=x$: $\frac{\sin(4x^2+8x^2)}{x^2+2x^2}=\frac{\sin(12x^2)}{3x^2}$

$\frac{\sin(4(x^2+2y^2))}{x^2+2y^2}=\frac{\sin 4u}{u}=4$

- F18E1#8



8. If

$$\lim_{(x,y)\to(0,0)} \frac{x^4-3a(x^2+y^2)-y^4}{x^2+y^2}=12,$$

then the number a must be equal to

A. 4
B. 6
C. 12
D. −4
E. 3

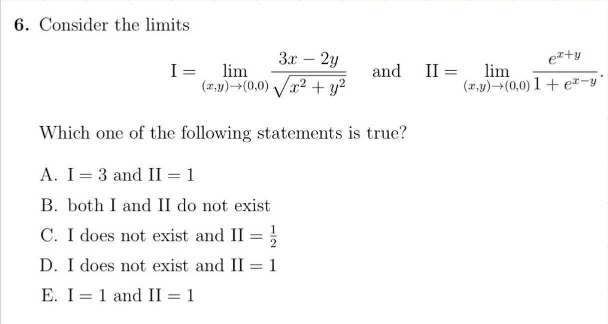
$\frac{x^4-y^4-3a(x^2+y^2)}{x^2+y^2}=12$

$\frac{\cancel{(x^2+y^2)}(x^2-y^2)}{\cancel{x^2+y^2}}-\frac{3a\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}}=12$

$(x^2-y^2)-3a=12$

$a=-4$

- S17E1#6



6. Consider the limits

$$\text{I} = \lim_{(x,y)\to(0,0)} \frac{3x-2y}{\sqrt{x^2+y^2}} \quad \text{and} \quad \text{II} = \lim_{(x,y)\to(0,0)} \frac{e^{x+y}}{1+e^{x-y}}.$$

Which one of the following statements is true?

A. I = 3 and II = 1
B. both I and II do not exist
C. I does not exist and II = $\frac{1}{2}$
D. I does not exist and II = 1
E. I = 1 and II = 1

I) $\frac{3(.5)-2(.5)}{\sqrt{(.5)^2+(.5)^2}} = \frac{1.5-1.0}{\sqrt{.5}}$

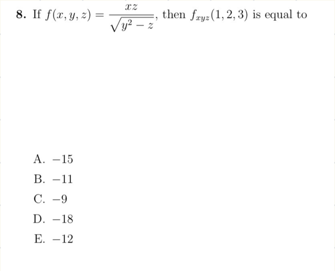
$\frac{3(-.5)-2(-.5)}{\sqrt{(-.5)^2+(-.5)^2}} = \frac{-1.5+1.0}{\sqrt{.5}}$

DNE

II) $\frac{e^0}{1+e^0} = \frac{1}{2}$

15.3 Partial Derivatives

- S19E1#8



8. If $f(x,y,z) = \frac{xz}{\sqrt{y^2-z}}$, then $f_{xyz}(1,2,3)$ is equal to

A. −15
B. −11
C. −9
D. −18
E. −12

$f_x = \frac{z}{\sqrt{y^2-z}} = z(y^2-z)^{-1/2}$

$f_{xy} = z\cdot(-\frac{1}{2})(2y)(y^2-z)^{-3/2}$

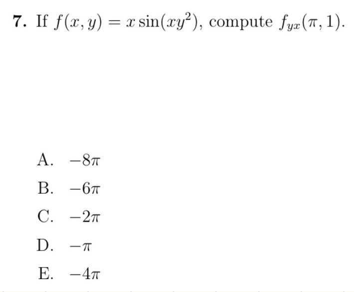
$= -yz(y^2-z)^{-3/2}$

$f_{xyz} = -yz(-3/2)(-1)(y^2-z)^{-5/2} + (-y)(y^2-z)^{-3/2}$

$f_{(1,2,3)} = -6(\frac{3}{2})(1) + (-2)(1)$

$= -9 - 2 = -11$

- S19FE#7



7. If $f(x,y) = x\sin(xy^2)$, compute $f_{yx}(\pi,1)$.

A. -8π
B. -6π
C. -2π
D. $-\pi$
E. -4π

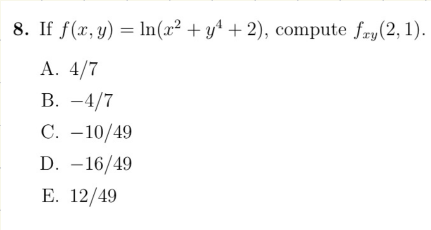
$f_y = 2xy\,x\cos(xy^2) = 2x^2y\cos(xy^2)$

$f_{yx} = 4xy\cos(xy^2) - \sin(xy^2)\,2x^2y^3$

$f_{(\pi,1)} = 4\pi\cdot\cos(\pi) - \sin(\pi)\,2\pi^2$

$= -4\pi$

• F19E1#8



8. If $f(x,y) = \ln(x^2 + y^4 + 2)$, compute $f_{xy}(2,1)$.

A. $4/7$

B. $-4/7$

C. $-10/49$

D. $-16/49$

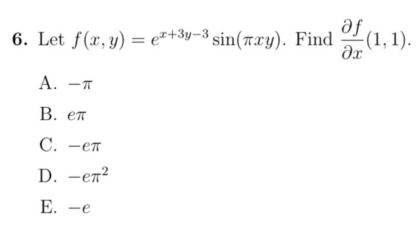
E. $12/49$

$$f_x = \frac{x^2}{x^2+y^4+2}$$

$$f_{xy} = \frac{-4y^3x^2}{(x^2+y^4+2)^2}$$

$$f_{(2,1)} = \frac{-4(1)(4)}{(4+1+2)^2} = \frac{-16}{49}$$

• F19FE#6



6. Let $f(x,y) = e^{x+3y-3}\sin(\pi xy)$. Find $\frac{\partial f}{\partial x}(1,1)$.

A. $-\pi$

B. $e\pi$

C. $-e\pi$

D. $-e\pi^2$

E. $-e$

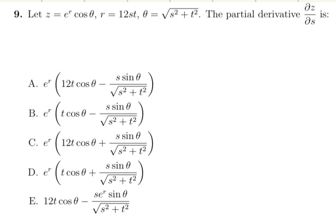
$$f_x = \sin(\pi xy)e^{x+3y-3} + \pi y\cos(\pi xy)e^{x+3y-3}$$

$$f_{(1,1)} = \sin(\pi)e + \pi\cos(\pi)e^1$$

$$= -\pi e$$

15.4 Chain Rule

• S19E1#9



9. Let $z = e^r\cos\theta$, $r = 12st$, $\theta = \sqrt{s^2+t^2}$. The partial derivative $\frac{\partial z}{\partial s}$ is:

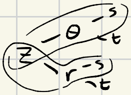
A. $e^r\left(12t\cos\theta - \frac{s\sin\theta}{\sqrt{s^2+t^2}}\right)$

B. $e^r\left(t\cos\theta - \frac{s\sin\theta}{\sqrt{s^2+t^2}}\right)$

C. $e^r\left(12t\cos\theta + \frac{s\sin\theta}{\sqrt{s^2+t^2}}\right)$

D. $e^r\left(t\cos\theta + \frac{s\sin\theta}{\sqrt{s^2+t^2}}\right)$

E. $12t\cos\theta - \frac{se^r\sin\theta}{\sqrt{s^2+t^2}}$



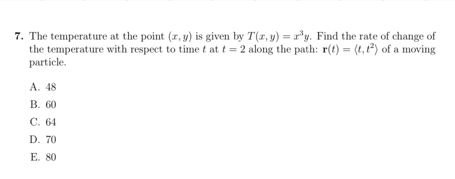
$$\frac{\partial z}{\partial s} = Z_\theta \cdot \Theta_s + Z_r \cdot R_s$$

$$= -e^r\sin\theta \cdot \tfrac{1}{2}(s^2+t^2)^{-1/2} \cdot 2s + e^r\cos\theta \cdot 12t$$

$$-e^r \frac{2s \cdot \sin\theta}{2\sqrt{s^2+t^2}} + e^r\, 12t\cos\theta$$

$$= er\left(12t\cos\theta - \frac{s \cdot \sin\theta}{\sqrt{s^2+t^2}}\right)$$

• F19FE#7



7. The temperature at the point (x,y) is given by $T(x,y) = x^3y$. Find the rate of change of the temperature with respect to time t at $t=2$ along the path: $\mathbf{r}(t) = \langle t, t^2\rangle$ of a moving particle.

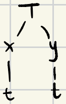
A. 48

B. 60

C. 64

D. 70

E. 80



$T = x^3y \qquad x = t \qquad y = t^2$

$$\frac{\partial T}{\partial t} = T_x \cdot x' + T_y \cdot y'$$

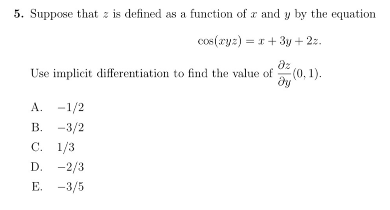
$$= 3x^2y(1) + x^3(2t)$$

$$= 3t^4 + t^3(2t)$$

$$= 3(16) + 8(4)$$

$$= 48 + 32 = 80$$

• S18FE #5



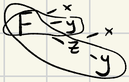
5. Suppose that z is defined as a function of x and y by the equation

$$\cos(xyz) = x + 3y + 2z.$$

Use implicit differentiation to find the value of $\frac{\partial z}{\partial y}(0, 1)$.

A. $-1/2$
B. $-3/2$
C. $1/3$
D. $-2/3$
E. $-3/5$

$F = x + 3y + 2z - \cos(xyz)$



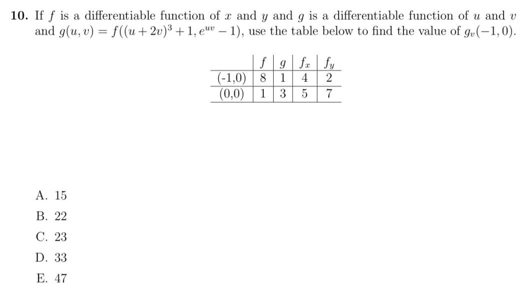
$F_y = 3 + xz\sin(xyz)$

$F_z = 2 + yx\sin(xyz)$

$F_y + F_z \cdot \frac{\partial z}{\partial y} = 0$

$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-3}{2}$

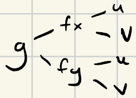
• F18E1 #10



10. If f is a differentiable function of x and y and g is a differentiable function of u and v and $g(u, v) = f((u + 2v)^3 + 1, e^{uv} - 1)$, use the table below to find the value of $g_v(-1, 0)$.

	f	g	f_x	f_y
(-1,0)	8	1	4	2
(0,0)	1	3	5	7

A. 15
B. 22
C. 23
D. 33
E. 47



$f(x,y)$ $(-1,0) \longrightarrow u = -1 \quad v = 0$

$(u+2v)^3 + 1 = x = -1+1 = 0$

$e^{uv} - 1 = y = 1-1 = 0$

Solving for x and y

$g_v(-1,0) = f_x X_v + f_y + Y_v$

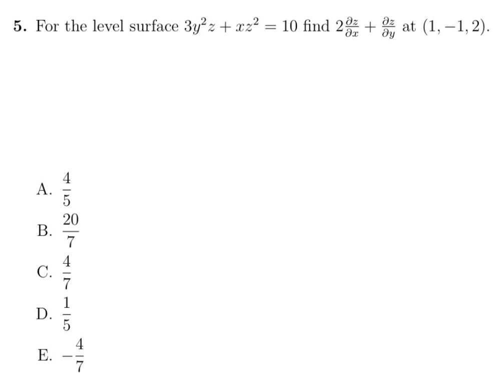
$= 5 X_v + 7 Y_v$

$X_v = 6(u+2v)^2$ —— plug $(-1,0) \longrightarrow 6$

$Y_v = ue^{uv}$ —— plug $(-1,0) \longrightarrow -1$

$g_v(-1,0) = 5(6) + 7(-1) = \boxed{23}$

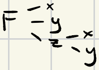
• F18FE #5



5. For the level surface $3y^2z + xz^2 = 10$ find $2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ at $(1, -1, 2)$.

A. $\frac{4}{5}$
B. $\frac{20}{7}$
C. $\frac{4}{7}$
D. $\frac{1}{5}$
E. $-\frac{4}{7}$

$F = 3y^2z + xz^2 - 10 = 0$



$F_x + F_z \cdot Z_x = 0$

$z^2 + [3y^2 + 2xz]\left(\frac{\partial z}{\partial x}\right) = 0$

$\frac{\partial z}{\partial x} = \frac{-z^2}{3y^2 + 2xz} = \frac{-4}{7}$

$F_y + F_z \cdot F_y = 0$

$6yz + 3y^2 + 2xz \cdot \frac{\partial z}{\partial y} = 0$

$\frac{\partial z}{\partial y} = \frac{-6yz}{3y^2 + 2xz} = \frac{12}{3+4} = \frac{12}{7}$

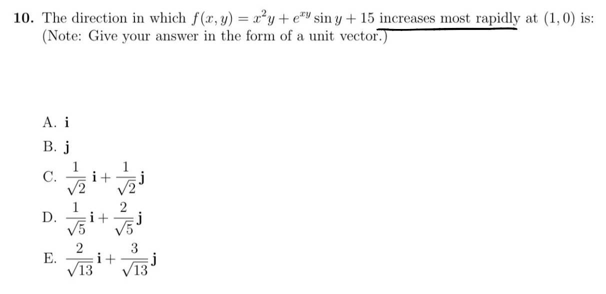
$2\left(\frac{-4}{7}\right) + \frac{12}{7} = \frac{4}{7}$

15.5 Directional Derivatives

· S19E1#10

$\frac{\nabla}{|\nabla|}$

$\nabla = \langle 2xy +$



10. The direction in which $f(x,y) = x^2y + e^{xy}\sin y + 15$ increases most rapidly at $(1,0)$ is: (Note: Give your answer in the form of a unit vector.)

A. $\mathbf{i}$

B. $\mathbf{j}$

C. $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

D. $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$

E. $\frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j}$

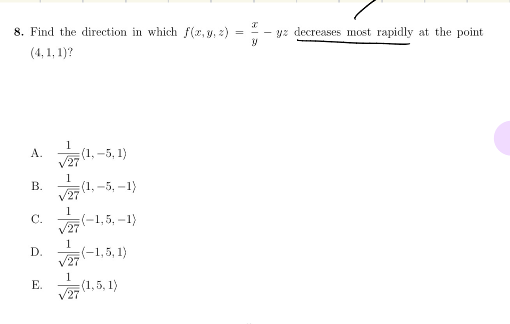
$\nabla = \langle 2xy + ye^{xy}\sin y,\ x^2 + \sin y \cdot xe^{xy} + \cos y e^{xy} \rangle$

$\nabla_{(1,0)} = \langle 0, 1+1 \rangle = \langle 0, 2 \rangle$

$\frac{\nabla}{|\nabla|} = \langle \frac{0}{2}, \frac{2}{2} \rangle = \langle 0, 1 \rangle = j$

· S19FE#8

$-\frac{\nabla}{|\nabla|}$



8. Find the direction in which $f(x,y,z) = \frac{x}{y} - yz$ decreases most rapidly at the point $(4,1,1)$?

A. $\frac{1}{\sqrt{27}}\langle 1,-5,1\rangle$

B. $\frac{1}{\sqrt{27}}\langle 1,-5,-1\rangle$

C. $\frac{1}{\sqrt{27}}\langle -1,5,-1\rangle$

D. $\frac{1}{\sqrt{27}}\langle -1,5,1\rangle$

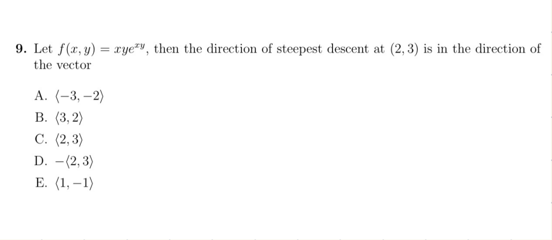
E. $\frac{1}{\sqrt{27}}\langle 1,5,1\rangle$

$\nabla = \langle \frac{1}{y}, -\frac{x}{y^2} - z, -y \rangle$

$\nabla_{(4,1,1)} = \langle 1, -5, -1 \rangle$

$-\frac{\nabla}{|\nabla|} = \langle -\frac{1}{\sqrt{27}}, \frac{+5}{\sqrt{27}}, \frac{+1}{\sqrt{27}} \rangle$

· F19E1#9



9. Let $f(x,y) = xye^{xy}$, then the direction of steepest descent at $(2,3)$ is in the direction of the vector

A. $\langle -3,-2\rangle$

B. $\langle 3,2\rangle$

C. $\langle 2,3\rangle$

D. $-\langle 2,3\rangle$

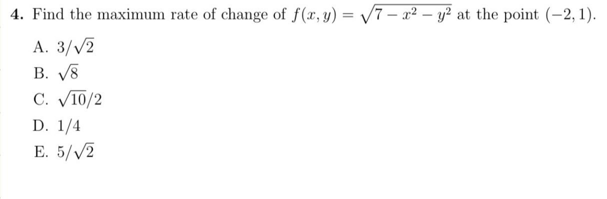
E. $\langle 1,-1\rangle$

$\nabla = \langle ye^{xy} + xy^2e^{xy},\ xe^{xy} + x^2ye^{xy} \rangle$

$\nabla_{(2,3)} = \langle 3e^6 + 18e^6,\ 2e^6 + 12e^6 \rangle = \langle 21e^6, 14e^6 \rangle$

$-\frac{\nabla}{|\nabla|} = \langle$

- F19FE#4



4. Find the maximum rate of change of $f(x,y) = \sqrt{7-x^2-y^2}$ at the point $(-2,1)$.

A. $3/\sqrt{2}$

B. $\sqrt{8}$

C. $\sqrt{10}/2$

D. $1/4$

E. $5/\sqrt{2}$

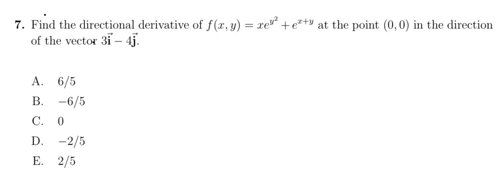
$f(x,y) = (7-x^2-y^2)^{1/2}$

$f_x = \frac{1}{2}(7-x^2-y^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{7-x^2-y^2}} = \frac{2}{\sqrt{7-4-1}} = \frac{2}{\sqrt{2}}$

$f_y = \frac{1}{2}(7-x^2-y^2)^{-1/2} \cdot (-2y) = \frac{-y}{\sqrt{7-x^2-y^2}} = \frac{1}{\sqrt{2}}$

$|\nabla| = \sqrt{\frac{4}{2} + \frac{1}{2}} = \sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{10}}{2}$

- S18FE#7



7. Find the directional derivative of $f(x,y) = xe^{y^2} + e^{x+y}$ at the point $(0,0)$ in the direction of the vector $3\vec{\mathbf{i}} - 4\vec{\mathbf{j}}$.

A. $6/5$

B. $-6/5$

C. 0

D. $-2/5$

E. $2/5$

$D = \nabla \cdot u$

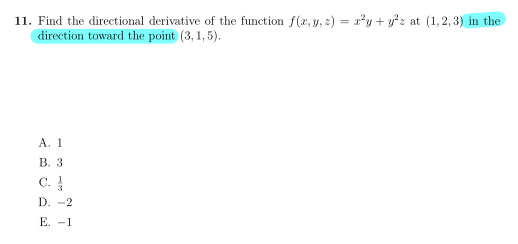
$u = \langle \frac{3}{5}, \frac{-4}{5} \rangle$

$\nabla = \langle e^{y^2} + e^{x+y}, 2yxe^{y^2} + e^{x+y})$

$\nabla_{(0,0)} = \langle e^0 + e^0, 0 + e^0 \rangle = \langle 2, 1 \rangle$

$D = \langle 2, 1 \rangle \cdot \langle \frac{3}{5}, \frac{-4}{5} \rangle = \frac{6}{5} - \frac{4}{5} = \frac{2}{5}$

- F18E1#11



11. Find the directional derivative of the function $f(x,y,z) = x^2y + y^2z$ at $(1,2,3)$ in the direction toward the point $(3,1,5)$.

A. 1

B. 3

C. $\frac{1}{3}$

D. -2

E. -1

$\nabla = \langle 2x, x^2 + 2yz, y^2 \rangle$

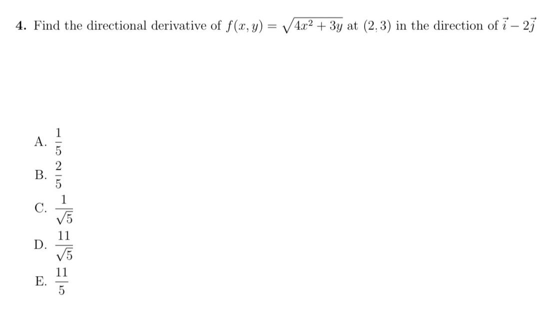
$V = \langle 2, -1, 2 \rangle$

$\nabla(1,2,3) = \langle 2, 13, 4 \rangle$

$\vec{u} = \langle \frac{2}{3}, \frac{-1}{3}, \frac{2}{3} \rangle$

$D = \langle 2, 13, 4 \rangle \cdot \langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \rangle = 1$

- F18 FE#4



4. Find the directional derivative of $f(x, y) = \sqrt{4x^2 + 3y}$ at $(2, 3)$ in the direction of $\vec{i} - 2\vec{j}$

A. $\frac{1}{5}$
B. $\frac{2}{5}$
C. $\frac{1}{\sqrt{5}}$
D. $\frac{11}{\sqrt{5}}$
E. $\frac{11}{5}$

$$u = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$$

$$\nabla = \left\langle \frac{1}{2}(4x^2+3y)^{-1/2} \cdot 8x, \frac{1}{2}(4x^2+3y)^{-1/2} \cdot 3 \right\rangle$$

$$= \left\langle \frac{4x}{\sqrt{4x^2+3y}}, \frac{3}{2\sqrt{4x^2+3y}} \right\rangle$$

$$\nabla_{(2,3)} = \left\langle \frac{8}{5}, \frac{3}{10} \right\rangle$$

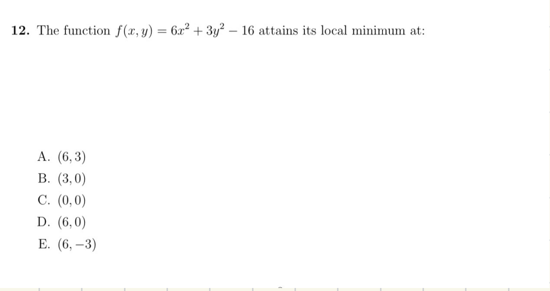
$$D = \nabla \cdot u = \left\langle \frac{16}{10}, \frac{3}{10} \right\rangle \cdot \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$$

$$= \frac{16}{10\sqrt{5}} + \frac{-6}{10\sqrt{5}} = \frac{10}{10\sqrt{5}} = \frac{1}{\sqrt{5}}$$

15.6 Tangent Plane and Linear Approximation

15.7 Max and Min Problems

• S19E1#12



12. The function $f(x, y) = 6x^2 + 3y^2 - 16$ attains its local minimum at:

A. (6, 3)
B. (3, 0)
C. (0, 0)
D. (6, 0)
E. (6, −3)

Min: $f_{xx} > 0$, $D > 0$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$f_x = 12x = 0$

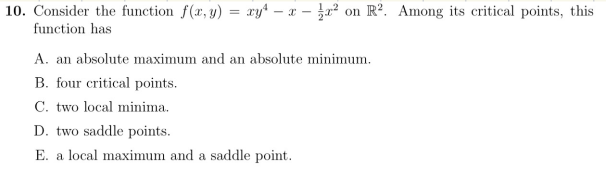
$f_y = 6y = 0$

$\rangle$ (0,0)

$f_{xx} = 12$ $f_{yy} = 6$ $f_{xy} = 0$

$D > 0$ $f_{xx} > 0$ at (0,0)

• F19E1#10



10. Consider the function $f(x, y) = xy^4 - x - \frac{1}{2}x^2$ on $\mathbb{R}^2$. Among its critical points, this function has

A. an absolute maximum and an absolute minimum.
B. four critical points.
C. two local minima.
D. two saddle points.
E. a local maximum and a saddle point.

① $f_x = y^4 - 1 - x = 0 \longrightarrow$ ③ $x = y^4 - 1 \rightarrow$ ② $4y^3(y^4-1) = 0$

② $f_y = 4y^3x = 0$

$$\underbrace{y = 0 \quad y = \pm 1}_{③}$$

$x = -1$ $x = 0$

CP: (-1,0), (0,1), (0,-1)

$f_{xx} = -1$

$f_{yy} = 12y^2$

$f_{xy} = 3y$

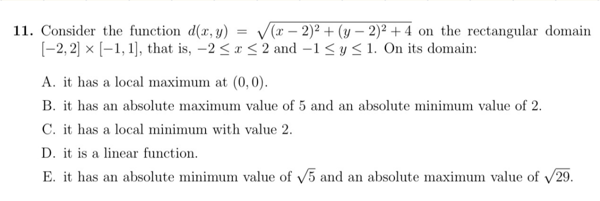
$D = -12y^2 - 9y^2$

$D(0,0) = 0$ inconclusive

$D(0,1) = -12 - 9 < 0$ Saddle

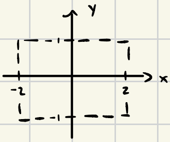
$D(0,-1) = -12 - 9 < 0$ Saddle

• F19E1#11



11. Consider the function $d(x, y) = \sqrt{(x-2)^2 + (y-2)^2 + 4}$ on the rectangular domain $[-2, 2] \times [-1, 1]$, that is, $-2 \le x \le 2$ and $-1 \le y \le 1$. On its domain:

A. it has a local maximum at (0, 0).
B. it has an absolute maximum value of 5 and an absolute minimum value of 2.
C. it has a local minimum with value 2.
D. it is a linear function.
E. it has an absolute minimum value of $\sqrt{5}$ and an absolute maximum value of $\sqrt{29}$.



$f_x: 2x - 4 = 0$ $x = 2$

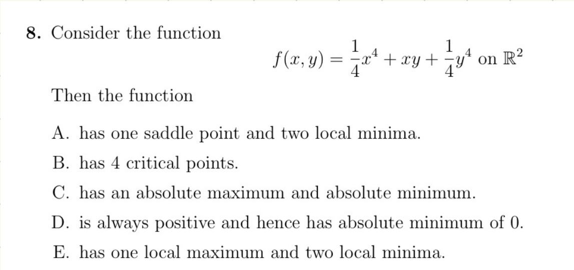
$f_y: 2y - 4 = 0$ $y = 2$

} (2,2) out of bounds

(-2,0), (2,0), (0,1), (0,-1)

$\downarrow$ $\sqrt{24}$, $\downarrow$ $\sqrt{4}$, $\downarrow$ $\sqrt{5}$, $\downarrow$ $\sqrt{13}$

· F19FE#8



8. Consider the function

$$f(x, y) = \frac{1}{4}x^4 + xy + \frac{1}{4}y^4 \text{ on } \mathbb{R}^2$$

Then the function

A. has one saddle point and two local minima.
B. has 4 critical points.
C. has an absolute maximum and absolute minimum.
D. is always positive and hence has absolute minimum of 0.
E. has one local maximum and two local minima.

$f_x = x^3 + y = 0 \longrightarrow y = -x^3$

$f_y = y^3 + x = 0 \longrightarrow (\overset{\downarrow}{-x^3})^3 + x = -x^9 + x = 0$

$x = \pm 1 \quad x = 0$

$y = \pm 1 \quad y = 0 \quad \Big\} \; (1,1), (-1,-1), (0,0)$

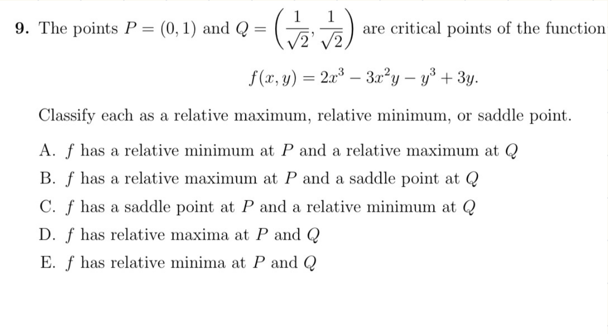
$f_{xx} = 3x^2 \qquad D = 9x^2y^2 - 1$

$f_{yy} = 3y^2 \qquad D(1,1) = 9 - 1 = 8 > 0 \qquad f_{xx} = 3 > 0$ min

$f_{xy} = 1 \qquad D(-1,-1) = 9 - 1 = 8 > 0 \qquad f_{xx} = 3 > 0$ Min

$D(0,0) = -1 < 0$ Saddle

· S18FE #9



9. The points $P = (0, 1)$ and $Q = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ are critical points of the function

$$f(x, y) = 2x^3 - 3x^2y - y^3 + 3y.$$

Classify each as a relative maximum, relative minimum, or saddle point.

A. f has a relative minimum at P and a relative maximum at Q
B. f has a relative maximum at P and a saddle point at Q
C. f has a saddle point at P and a relative minimum at Q
D. f has relative maxima at P and Q
E. f has relative minima at P and Q

$f_x = 6x^2 - 6xy \qquad D = (12x - 6y)(-6y) - (-6x)^2$

$f_y = -3x^2 - 3y^2 + 3 \qquad D(0,1) = -6(-6) = 12 \qquad f_{xx}(0,1) = -6$ P: Max

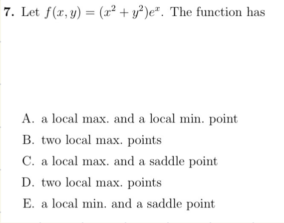
$f_{xx} = 12x - 6y \qquad D(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \left(\frac{12}{\sqrt{2}} - \frac{6}{\sqrt{2}}\right)\left(\frac{-6}{\sqrt{2}}\right) - \left(\frac{-6}{\sqrt{2}}\right)^2$

$= \left(\frac{-36}{2}\right) - \left(\frac{36}{2}\right)$ Q: Saddle

$f_{yy} = -6y$

$f_{xy} = -6x$

· F18FE#7



7. Let $f(x, y) = (x^2 + y^2)e^x$. The function has

A. a local max. and a local min. point
B. two local max. points
C. a local max. and a saddle point
D. two local max. points
E. a local min. and a saddle point

$f_x = e^x(2x) + e^x(x^2+y^2) = 0$

$f_y = 2ye^x = 0 \rightarrow y = 0 \longrightarrow e^x(2x + x^2) = 0$

$x = 0 \quad x = -2$

CP: $(0,0), (-2,0)$

$f_{xx} = e^x(y+2+4x+x^2)$

$f_{xy} = 2e^{xy}$

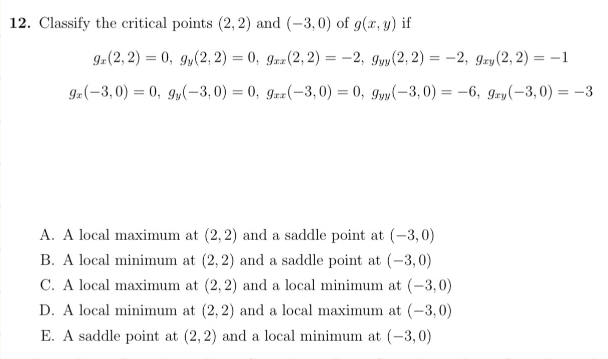
$f_{yy} = 2e^x$

$D = 2e^x(e^x)(y+2+4x+x^2) - (2e^{xy})^2$

$D(0,0) > 0 \quad f_{xx} < 0$ min

$D(-2,0) < 0$ saddle

· F18E1#12



12. Classify the critical points $(2, 2)$ and $(-3, 0)$ of $g(x, y)$ if

$$g_x(2,2) = 0,\ g_y(2,2) = 0,\ g_{xx}(2,2) = -2,\ g_{yy}(2,2) = -2,\ g_{xy}(2,2) = -1$$

$$g_x(-3,0) = 0,\ g_y(-3,0) = 0,\ g_{xx}(-3,0) = 0,\ g_{yy}(-3,0) = -6,\ g_{xy}(-3,0) = -3$$

A. A local maximum at $(2, 2)$ and a saddle point at $(-3, 0)$
B. A local minimum at $(2, 2)$ and a saddle point at $(-3, 0)$
C. A local maximum at $(2, 2)$ and a local minimum at $(-3, 0)$
D. A local minimum at $(2, 2)$ and a local maximum at $(-3, 0)$
E. A saddle point at $(2, 2)$ and a local minimum at $(-3, 0)$

$D = g_{xx}g_{yy} - g_{xy}^2$

$-2(-2) - (-1)^2 = 3 > 0$

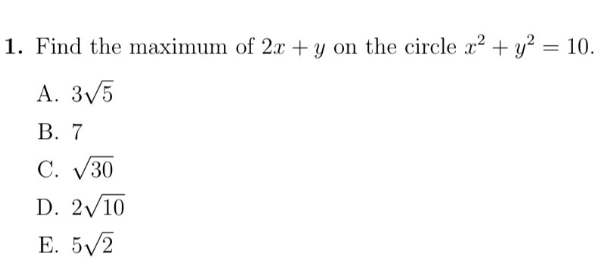
$g_{xx} < 0$ max $(2,2)$

$D = g_{xx}g_{yy} - g_{xy}^2$

$0(-6) - (-3)^2 = -9 < 0$ saddle $(-3,0)$

15.8 Lagrange Multipliers

· S18E2#1



1. Find the maximum of $2x + y$ on the circle $x^2 + y^2 = 10$.

A. $3\sqrt{5}$
B. 7
C. $\sqrt{30}$
D. $2\sqrt{10}$
E. $5\sqrt{2}$

$F = 2x+y \quad g = x^2+y^2-10 \quad \nabla F = \langle 2, 1\rangle \quad \nabla g = \langle 2x, 2y\rangle$

$\langle 2, 1\rangle = \lambda\langle 2x, 2y\rangle$

① $2 = \lambda 2x$ ② $1 = \lambda 2y$

$0 = \lambda 2x - 2 \qquad 0 = \lambda 2y - 1$

$\lambda = \frac{2}{2x} \qquad \lambda = \frac{1}{2y}$

$x = 2y$

$\hookrightarrow g(x,y) = 4y^2+y^2-10 = 0$

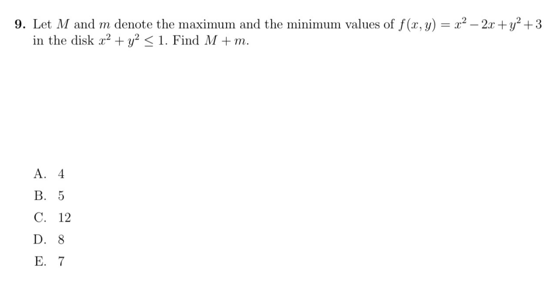
$y = \pm\sqrt{2}$

CP: $\overset{③}{(2\sqrt{2}, \sqrt{2})}, \overset{④}{(-2\sqrt{2}, -\sqrt{2})}$

$F_{③} = 4\sqrt{2} + \sqrt{2} = 5\sqrt{2}$

$F_{④} = -4\sqrt{2} - \sqrt{2} = -5\sqrt{2}$

· S19FE#9



9. Let M and m denote the maximum and the minimum values of $f(x, y) = x^2 - 2x + y^2 + 3$ in the disk $x^2 + y^2 \leq 1$. Find $M + m$.

A. 4
B. 5
C. 12
D. 8
E. 7

1) Write equation $\nabla F = \lambda \nabla g$
2) Solve for λ and set equations equal to one another
3) Solve for $x = y$ format and plug into $g(x,y)$
4) Solve for x or y then plug back into Step 4 equation
5) Plug point into $F(x,y)$

$F = x^2 - 2x + y^2 + 3$ $\quad g(x,y) = x^2 + y^2 - 1$

$\nabla F = \langle 2x - 2, 2y \rangle$ $\quad \nabla g = \langle 2x, 2y \rangle$

① $2x - 2 = \lambda 2x$

$2x - 2 - \lambda 2x = 0$

$2x(1 - \lambda) = 2$

$x = 1$ $\quad \lambda = -1$

$\downarrow$ $g(x,y)$: $y = 0$

$\downarrow$ ②: $2y = -2y$, $4y = 0$, $y = 0$ $\downarrow$ $g(x,y)$ $\downarrow$ $x = \pm 1$

② $2y = \lambda 2y$

$2y - \lambda 2y = 0$

$2y(1 - \lambda) = 0$

$y = 0$ $\quad \lambda = 1$

$\downarrow$ $g(x,y)$: $x = \pm 1$

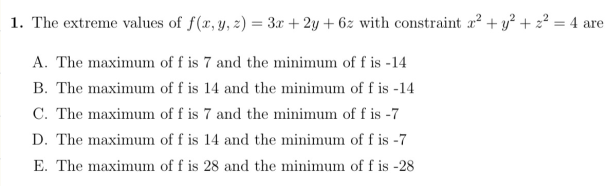
CP: $(1,0)$, $(-1,0)$

$f(1,0) = 1 - 2 + 3 = 2$

$f(-1,0) = (-1)^2 + 2 + 3 = 6$

$M + m = 6 + 2 = 8$

· S19E2#1



1. The extreme values of $f(x, y, z) = 3x + 2y + 6z$ with constraint $x^2 + y^2 + z^2 = 4$ are

A. The maximum of f is 7 and the minimum of f is -14
B. The maximum of f is 14 and the minimum of f is -14
C. The maximum of f is 7 and the minimum of f is -7
D. The maximum of f is 14 and the minimum of f is -7
E. The maximum of f is 28 and the minimum of f is -28

$\nabla F = \langle 3, 2, 6 \rangle$ $\quad \nabla g = \langle 2x, 2y, 2z \rangle$

$3 = \lambda 2x$ $\quad 2 = \lambda 2y$ $\quad 6 = \lambda 2z$

$\lambda = \frac{3}{2x}$ $\quad \lambda = \frac{2}{2y}$ $\quad \lambda = \frac{6}{2z}$

$$\frac{3}{2x} = \frac{2}{2y} = \frac{6}{2z} \rightarrow \frac{3}{2x} = \frac{1}{y} = \frac{3}{z}$$

① $x = \frac{3y}{2}$ $\quad 3y = z$ ② $\rightarrow g(x,y,z)$

$g(x,y,z) = \left(\frac{3}{2}y^2\right) + y^2 + (3y^2) = 4$

$\frac{9}{4}y^2 + y^2 + 9y^2 = 4$

$49y^2 = 16$

$y = \pm\frac{4}{7}$ (plug into ①, ②)

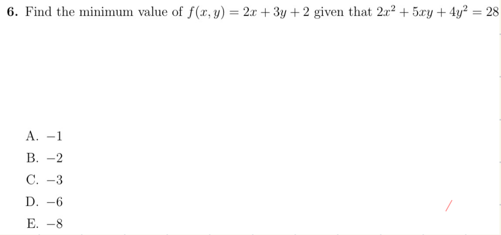
$x = \pm\frac{6}{7}$ $\quad z = \pm\frac{12}{7}$

$\left(\frac{6}{7}, \frac{4}{7}, \frac{12}{7}\right) \rightarrow f(x,y,z) =$ 14 max

$\left(-\frac{6}{7}, -\frac{4}{7}, -\frac{12}{7}\right) \rightarrow f(x,y,z) =$ -14 min

We know this b/c shape is a square defined by constant (Find intercepts of each)

- F18 FE#6



6. Find the minimum value of $f(x,y) = 2x + 3y + 2$ given that $2x^2 + 5xy + 4y^2 = 28$

A. -1
B. -2
C. -3
D. -6
E. -8

$\nabla F = \langle 2, 3 \rangle \qquad \nabla g = \langle 4x + 5y,\ 5x + 8y \rangle$

$2 = \lambda(4x + 5y)$

$\lambda = \frac{2}{(4x+5y)}$

$3 = \lambda(5x + 8y)$

$\lambda = \frac{3}{(5x+8y)}$

$\frac{2}{(4x+5y)} = \frac{3}{(5x+8y)}$

$3(4x+5y) = 2(5x+8y)$

$12x + 15y = 10x + 16y$

$2x = y \longrightarrow$

$g(x,y) = 2x^2 + 5x(2x) + 4(2x)^2 = 28$

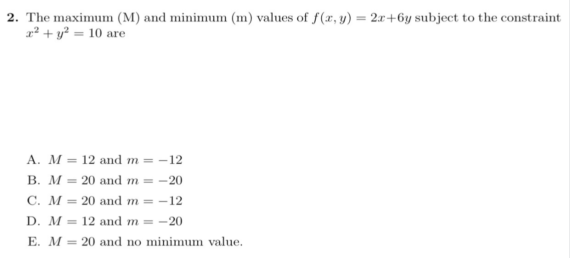
$18x^2 + 10x^2 = 28$

$x = \pm 1$

$(1, 2), (-1, -2)$

$F(-1,-2) = 2(-1) + 3(-2) + 2 = -6$

- F18E2#2



2. The maximum (M) and minimum (m) values of $f(x,y) = 2x + 6y$ subject to the constraint $x^2 + y^2 = 10$ are

A. $M = 12$ and $m = -12$
B. $M = 20$ and $m = -20$
C. $M = 20$ and $m = -12$
D. $M = 12$ and $m = -20$
E. $M = 20$ and no minimum value.

$\nabla F = \langle 2, 6 \rangle \qquad \nabla g = \langle 2x, 2y \rangle$

$2 = \lambda 2x \qquad 6 = \lambda 2y$

$\lambda = \frac{2}{2x} \qquad \lambda = \frac{6}{2y}$

$\frac{2}{2x} = \frac{6}{2y}$

$4y = 12x$

$y = 3x \longrightarrow g(x,y) = x^2 + (3x)^2 - 10 = 0$

$x^2 + 9x^2 - 10 = 0$

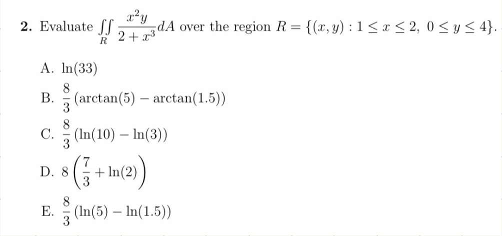
$x = \pm 1$

$(-1,-3), (1,3) \longrightarrow F(-1,-3) = -2 - 18 = -20$

$F(1,3) = 2 + 18 = 20$

16.1 Double Integrals in Rectangular Regions

· F19E2#2



2. Evaluate $\iint\limits_R \frac{x^2y}{2+x^3}dA$ over the region $R = \{(x,y) : 1 \le x \le 2,\ 0 \le y \le 4\}$.

A. ln(33)

B. $\frac{8}{3}$(arctan(5) − arctan(1.5))

C. $\frac{8}{3}$(ln(10) − ln(3))

D. $8\left(\frac{7}{3} + \ln(2)\right)$

E. $\frac{8}{3}$(ln(5) − ln(1.5))

$$\int_0^4 \int_1^2 \frac{x^2y}{2+x^3}\,dx\,dy$$

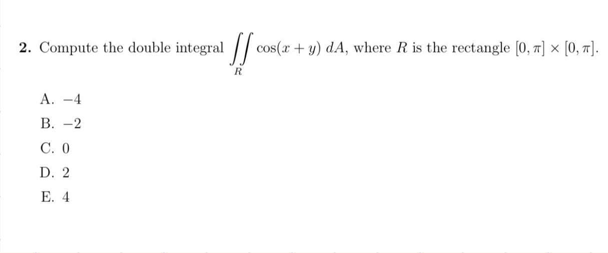
$u = 2+x^3$

$du = 3x^2$

$$\int_0^4 \frac{y}{3}\int_3^{10} \frac{1}{u}\,dx\,dy = \int_0^4 \frac{y}{3}\left(\ln(10) - \ln(3)\right) =$$

$$= \frac{y^2}{6}\left[\ln(10) - \ln(3)\right]\Big|_0^4 = \frac{8}{3}\left[\ln(10) - \ln(3)\right]$$

· S18E2#2



2. Compute the double integral $\iint\limits_R \cos(x+y)\,dA$, where R is the rectangle $[0,\pi] \times [0,\pi]$.

A. −4

B. −2

C. 0

D. 2

E. 4

$$\int_0^\pi \int_0^\pi \cos(x+y)\,dx\,dy$$

$u = x+y$

$du = 1$

$$\int_0^\pi \int_y^{\pi+y} \cos(u)\,du\,dy = \int_0^\pi \sin(u)\Big|_y^{\pi+y} dy = \int_0^\pi [\sin(\pi+y) - \sin(y)]\,dy$$

$$= \int_0^\pi \sin(\pi+y) - \int_0^\pi \sin(y)$$

$u = \pi+y$

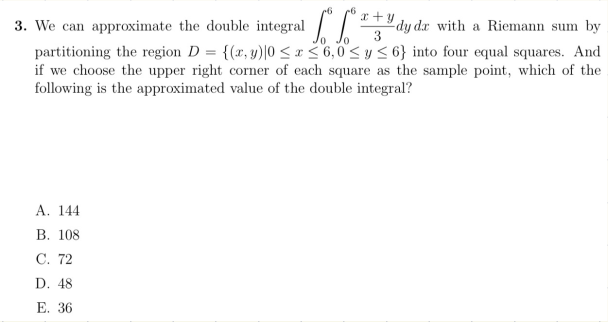
$du = 1$

$$\int_\pi^{2\pi} \sin(u)\,du - \int_0^\pi \sin(y)\,dy$$

$$-\cos(u)\Big|_\pi^{2\pi} + \cos(y)\Big|_0^\pi$$

$$-(1+1) + (-1-1) = -4$$

· F18E2#3



3. We can approximate the double integral $\int_0^6 \int_0^6 \frac{x+y}{3}dy\,dx$ with a Riemann sum by partitioning the region $D = \{(x,y) | 0 \le x \le 6, 0 \le y \le 6\}$ into four equal squares. And if we choose the upper right corner of each square as the sample point, which of the following is the approximated value of the double integral?

A. 144

B. 108

C. 72

D. 48

E. 36

$$\int_0^6 \int_0^6 \frac{x+y}{3}\,dy\,dx$$

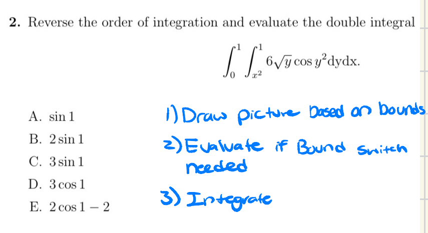
$$\int_0^6 \frac{1}{3}\left(xy + \frac{1}{2}y^2\right)\Big|_0^6 dx = \int_0^6 \frac{1}{3}(6x + 18)\,dx$$

$$= \int_0^6 (2x+6)\,dx$$

$$x^2 + 6x\Big|_0^6 = 36 + 36 = 108$$

16.2 Double Integrals over General regions

- S19 E2 #2

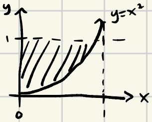


2. Reverse the order of integration and evaluate the double integral

$$\int_0^1\int_{x^2}^1 6\sqrt{y}\cos y^2 dydx.$$

A. sin 1
B. 2 sin 1
C. 3 sin 1
D. 3 cos 1
E. 2 cos 1 − 2

1) Draw picture based on bounds
2) Evaluate if Bound switch needed
3) Integrate

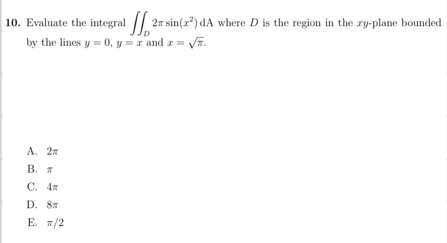


$$\underset{y}{\int_0^1}\underset{x}{\int_0^{\sqrt{y}}} 6\sqrt{y}\cos y^2\,dx\,dy = \int_0^1 x6\sqrt{y}\cos y^2\Big|_0^{\sqrt{y}} dy$$

$$\int_0^1 6y\cos y^2\,dy \qquad \begin{matrix} u=y^2 \\ du=2y \end{matrix}$$

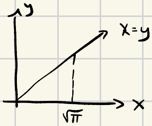
$$3\int_0^1 \cos(u)\,du = 3(\sin(1))$$

- S19 FE #10



10. Evaluate the integral $\iint_D 2\pi\sin(x^2)\,dA$ where D is the region in the xy-plane bounded by the lines $y = 0$, $y = x$ and $x = \sqrt{\pi}$.

A. 2π
B. π
C. 4π
D. 8π
E. $\pi/2$



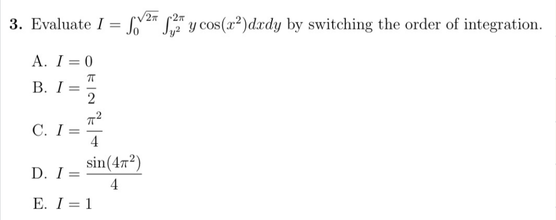
$$\int_0^{\sqrt{\pi}}\int_0^x 2\pi\sin(x^2)\,dy\,dx$$

$$\int_0^{\sqrt{\pi}} x2\pi\sin(x^2)\,dx$$

$u = x^2 \quad du = 2x$

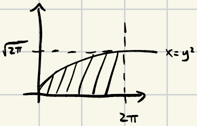
$$\pi\int_0^{\pi}\sin(u)\,du = -\pi\cos(u)\Big|_0^{\pi} = \pi+\pi = 2\pi$$

- F19 E2 #3



3. Evaluate $I = \int_0^{\sqrt{2\pi}}\int_{y^2}^{2\pi} y\cos(x^2)dxdy$ by switching the order of integration.

A. $I = 0$
B. $I = \dfrac{\pi}{2}$
C. $I = \dfrac{\pi^2}{4}$
D. $I = \dfrac{\sin(4\pi^2)}{4}$
E. $I = 1$



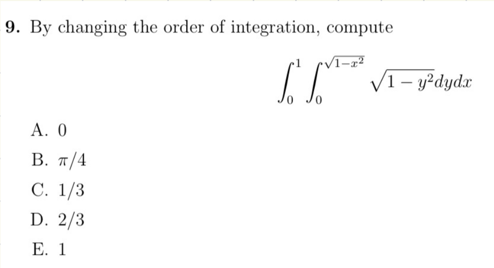
$$\int_0^{2\pi}\int_0^{\sqrt{x}} y\cos(x^2)\,dy\,dx$$

$$\int_0^{2\pi}\cos(x^2)\left[\tfrac{1}{2}y^2\Big|_0^{\sqrt{x}}\right]dx$$

$$\int_0^{2\pi}\cos(x^2)\,\frac{x}{2}\,dx = \frac{1}{4}\int_0^{4\pi^2}\cos(u) = \frac{\sin(4\pi^2)}{4}$$

$u = x^2 \quad du = 2x$

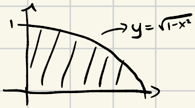
- F19 FE #9



9. By changing the order of integration, compute

$$\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-y^2}dydx$$

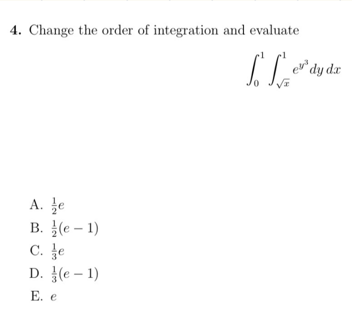
A. 0
B. $\pi/4$
C. 1/3
D. 2/3
E. 1



$$\int_0^1\int_0^{\sqrt{-y^2+1}}\sqrt{1-y^2}\,dx\,dy$$

$$\int_0^1(1-y^2)dy = y-\tfrac{1}{3}y^3\Big|_0^1 = 1-\tfrac{1}{3} = \tfrac{2}{3}$$

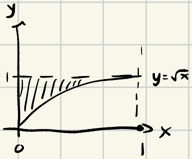
· F18E2# 4



4. Change the order of integration and evaluate

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\,dy\,dx$$

A. $\frac{1}{2}e$
B. $\frac{1}{2}(e-1)$
C. $\frac{1}{3}e$
D. $\frac{1}{3}(e-1)$
E. e

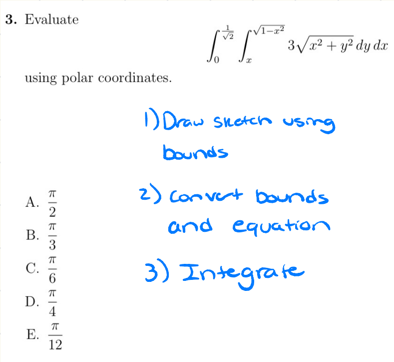


$\int_0^1 \int_0^{y^2} e^{y^3}\,dx\,dy$

$\int_0^1 y^2 e^{y^3}$ $u=y^3$ $du=\frac{1}{3}y^2$ $\frac{1}{3}\int_0^1 e^u\,du = \frac{1}{3}(e-1)$

16.3 Double Integrals in Polar Coordinates

· S19E2 #3



3. Evaluate

$$\int_0^{\frac{1}{\sqrt{2}}} \int_x^{\sqrt{1-x^2}} 3\sqrt{x^2+y^2}\,dy\,dx$$

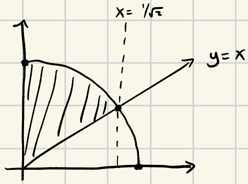
using polar coordinates.

1) Draw sketch using bounds

2) convert bounds and equation

3) Integrate

A. $\frac{\pi}{2}$
B. $\frac{\pi}{3}$
C. $\frac{\pi}{6}$
D. $\frac{\pi}{4}$
E. $\frac{\pi}{12}$



$0 \le r \le 1$

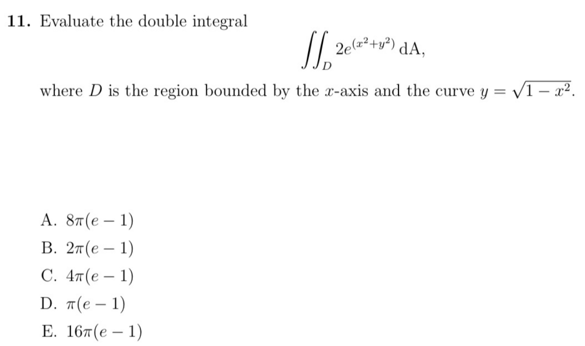
$\frac{\pi}{4} \le \theta \le \pi/2$

$\frac{\sqrt{2}}{2} = 45°$

$$\int_{\pi/4}^{\pi/2} \int_0^1 3r^2\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} r^3\Big|_0^1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

· S19FE # 11

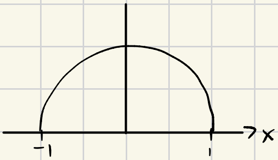


11. Evaluate the double integral

$$\iint_D 2e^{(x^2+y^2)}\,\mathrm{dA},$$

where D is the region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.

A. $8\pi(e-1)$
B. $2\pi(e-1)$
C. $4\pi(e-1)$
D. $\pi(e-1)$
E. $16\pi(e-1)$

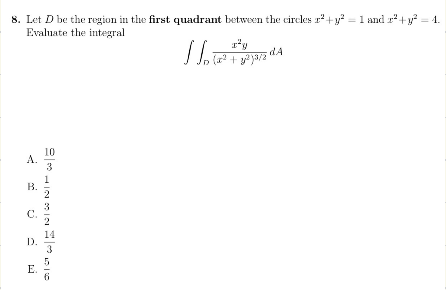


$\int_0^{\pi}\int_0^1 2re^{r^2}\,dr\,d\theta = \int_0^{\pi}\int_0^1 e^u\,du\,d\theta$

$u=r^2$ $du=2r$ $= \int_0^{\pi}(e-1)$

$= \pi(e-1)$

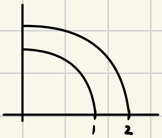
- F18FE#8



8. Let D be the region in the **first quadrant** between the circles $x^2+y^2=1$ and $x^2+y^2=4$. Evaluate the integral

$$\iint_D \frac{x^2y}{(x^2+y^2)^{3/2}}\,dA$$

A. $\frac{10}{3}$
B. $\frac{1}{2}$
C. $\frac{3}{2}$
D. $\frac{14}{3}$
E. $\frac{5}{6}$



$x = r\cos\theta$
$y = r\sin\theta$

$1 \le r \le 2$

$0 \le \theta \le \pi/2$

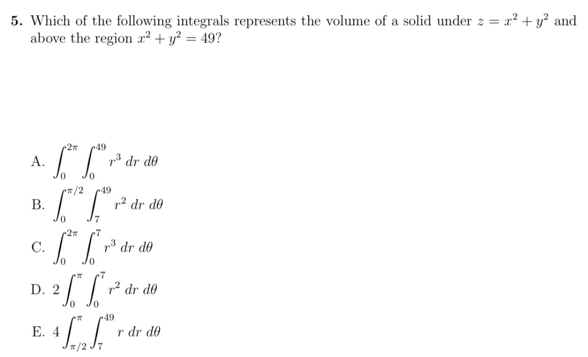
$$\int_0^{\pi/2}\int_1^2 \frac{r^3\cos\theta\sin\theta}{(r^2)^{3/2}}\,dr\,d\theta$$

$$\int_0^{\pi/2}\int_1^2 \cos\theta\sin\theta\,dr\,d\theta$$

$$\int_0^{\pi/2}(\cos\theta\sin\theta)\,d\theta = \int_0^1 u\,du = \frac{1}{2}u^2 = \frac{1}{2}$$

$u = \sin\theta$
$du = \cos\theta$

- F18E2 #5



5. Which of the following integrals represents the volume of a solid under $z = x^2+y^2$ and above the region $x^2+y^2 = 49$?

A. $\int_0^{2\pi}\int_0^{49} r^3\,dr\,d\theta$
B. $\int_0^{\pi/2}\int_7^{49} r^2\,dr\,d\theta$
C. $\int_0^{2\pi}\int_0^{7} r^3\,dr\,d\theta$
D. $2\int_0^{\pi}\int_0^{7} r^2\,dr\,d\theta$
E. $4\int_{\pi/2}^{\pi}\int_7^{49} r\,dr\,d\theta$

$x^2+y^2 = r^2 = 49$ $\quad$ $x^2+y^2 = z$ $\quad$ $z = 49$

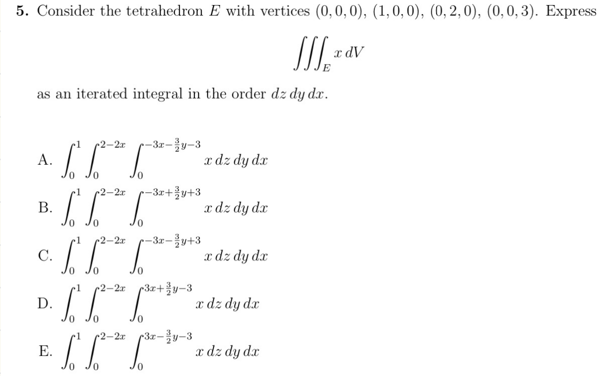
$r = 7$

$0 \le r \le 7$

$$\int_0^{2\pi}\int_0^7 r^3\,dr\,d\theta$$

16.4 Triple Integrals

- S19E2 #5

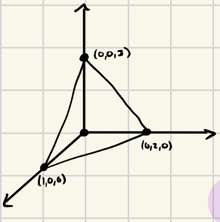


5. Consider the tetrahedron E with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$. Express

$$\iiint_E x\,dV$$

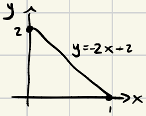
as an iterated integral in the order $dz\,dy\,dx$.

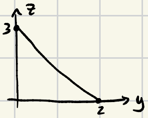
A. $\int_0^1\int_0^{2-2x}\int_0^{-3x-\frac{3}{2}y-3} x\,dz\,dy\,dx$
B. $\int_0^1\int_0^{2-2x}\int_0^{-3x+\frac{3}{2}y+3} x\,dz\,dy\,dx$
C. $\int_0^1\int_0^{2-2x}\int_0^{-3x-\frac{3}{2}y+3} x\,dz\,dy\,dx$
D. $\int_0^1\int_0^{2-2x}\int_0^{3x+\frac{3}{2}y-3} x\,dz\,dy\,dx$
E. $\int_0^1\int_0^{2-2x}\int_0^{3x-\frac{3}{2}y-3} x\,dz\,dy\,dx$

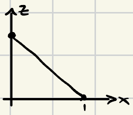


$$\int_0^1\int_0^{2-2x}\int_0^{3-3x-3/2y} x\,dz\,dy\,dx$$

$0 \le x \le 1$





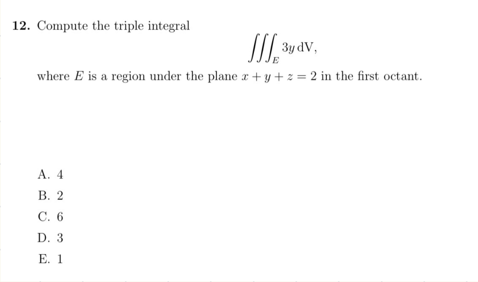


$0 \le y \le -2x+2$

$z = -\frac{3}{2}y+3$ $\quad$ $z = -3x+3$

$$\underbrace{\qquad\qquad\qquad}_{0 \le z \le 3-3x-\frac{3}{2}y}$$

- S19FE#12

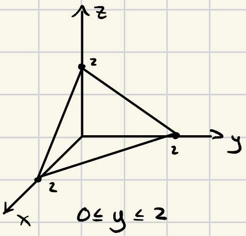


12. Compute the triple integral

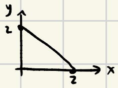
$$\iiint_E 3y\,\mathrm{dV},$$

where E is a region under the plane $x+y+z=2$ in the first octant.

A. 4
B. 2
C. 6
D. 3
E. 1



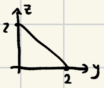
$0 \le y \le 2$



$y = -2x + 2$

$0 \le x \le 1 - \frac{y}{2}$

$0 \le x \le 2 - y$



$2z = -2x \qquad 2z - 2y$

$0 \le z \le 2 - x - y$

$$\int_0^2 \int_0^{2-y} \int_0^{2-x-y} 3y\,dz\,dx\,dy$$

$$\int_0^2 \int_0^{2-y} 3y(2-x-y)\,dx\,dy$$

$$6xy - \frac{3}{2}yx^2 - 3y^2x \Big|_0^{2-y}$$

$$6y(2-y) - \frac{3}{2}y(2-y)^2 - 3y^2(2-y)$$

$$12y - 6y^2 - \frac{3}{2}y(4-4y+y^2) - 6y^2 + 3y^3$$

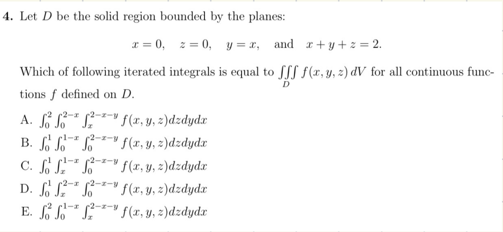
$$3y^3 - 12y^2 + 12y - \frac{12y}{2} + \frac{12y^2}{2} - \frac{3y^3}{2}$$

$$12y - 6y^2 - 6y + 6y^2 - \frac{3}{2}y^3 - 6y^2 + 3y^3$$

$$6y^2 - \cancel{2y^3} - 3y^2 + \cancel{2y^3} - \frac{3}{8}y^4 - 2y^3 + \frac{3}{4}y^4 \Big|_0^2$$

$$3y^2 - \frac{3}{8}y^4 - 2y^3 \Big|_0^2 = 12 - \frac{16\cdot 3}{8} - 16 = 2$$

- F19E2#4

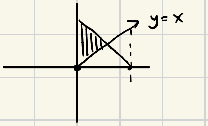


4. Let D be the solid region bounded by the planes:

$$x=0, \quad z=0, \quad y=x, \quad \text{and} \quad x+y+z=2.$$

Which of following iterated integrals is equal to $\iiint\limits_D f(x,y,z)\,dV$ for all continuous functions f defined on D.

A. $\int_0^2 \int_0^{2-x} \int_x^{2-x-y} f(x,y,z)dzdydx$
B. $\int_0^1 \int_0^{1-x} \int_0^{2-x-y} f(x,y,z)dzdydx$
C. $\int_0^1 \int_x^{1-x} \int_0^{2-x-y} f(x,y,z)dzdydx$
D. $\int_0^1 \int_x^{2-x} \int_0^{2-x-y} f(x,y,z)dzdydx$
E. $\int_0^2 \int_0^{1-x} \int_x^{2-x-y} f(x,y,z)dzdydx$

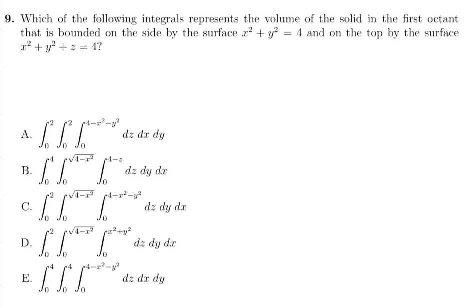


$x + y = 2$
$y = 2 - x$

$0 \le x \le 1 \qquad 0 \le y \le 2 - x$

$$\int_0^1 \int_0^{2-x} \int_0^{2-x-y} f(x,y,z)\,dz\,dy\,dx$$

- F18FE#9



9. Which of the following integrals represents the volume of the solid in the first octant that is bounded on the side by the surface $x^2+y^2=4$ and on the top by the surface $x^2+y^2+z=4$?

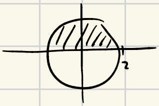
A. $\displaystyle\int_0^2 \int_0^2 \int_0^{4-x^2-y^2} dz\,dx\,dy$

B. $\displaystyle\int_0^4 \int_0^{\sqrt{4-x^2}} \int_0^{4-z} dz\,dy\,dx$

C. $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} dz\,dy\,dx$

D. $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{x^2+y^2} dz\,dy\,dx$

E. $\displaystyle\int_0^4 \int_0^4 \int_0^{4-x^2-y^2} dz\,dx\,dy$

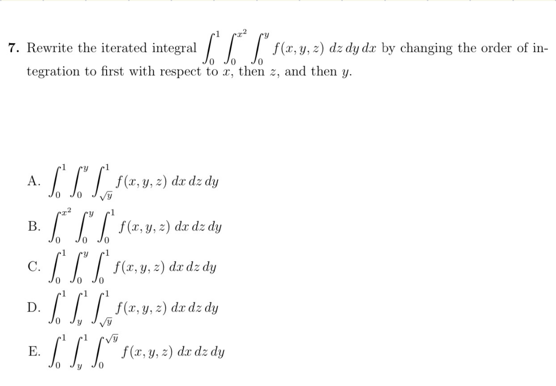


$0 \le x \le 2$

$0 \le y \le \sqrt{4-x^2}$

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} dz\,dy\,dx$$

· F18E2#7



7. Rewrite the iterated integral $\int_0^1\int_0^{x^2}\int_0^{y} f(x,y,z)\,dz\,dy\,dx$ by changing the order of integration to first with respect to x, then z, and then y.

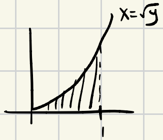
A. $\int_0^1\int_0^y\int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$

B. $\int_0^{x^2}\int_0^y\int_0^1 f(x,y,z)\,dx\,dz\,dy$

C. $\int_0^1\int_0^y\int_0^1 f(x,y,z)\,dx\,dz\,dy$

D. $\int_0^1\int_y^1\int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$

E. $\int_0^1\int_y^1\int_0^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy$



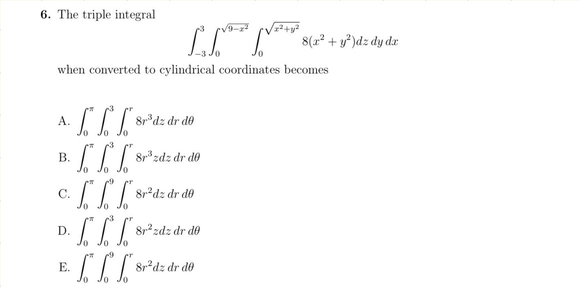
$\sqrt{y} \le x \le 1$

$0 \le y \le 1$

$\int_0^1 \int_0^y \int_{\sqrt{y}}^1 dx\,dz\,dy$

16.5 Triple Integrals in Cylindrical and Spherical Coordinates

· S19E2#6



6. The triple integral

$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{x^2+y^2}} 8(x^2+y^2)dz\,dy\,dx$$

when converted to cylindrical coordinates becomes

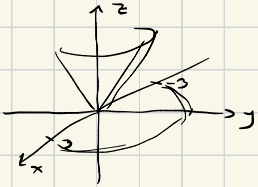
A. $\int_0^\pi\int_0^3\int_0^r 8r^3 dz\,dr\,d\theta$

B. $\int_0^\pi\int_0^3\int_0^r 8r^3 z dz\,dr\,d\theta$

C. $\int_0^\pi\int_0^9\int_0^r 8r^2 dz\,dr\,d\theta$

D. $\int_0^\pi\int_0^3\int_0^r 8r^2 z dz\,dr\,d\theta$

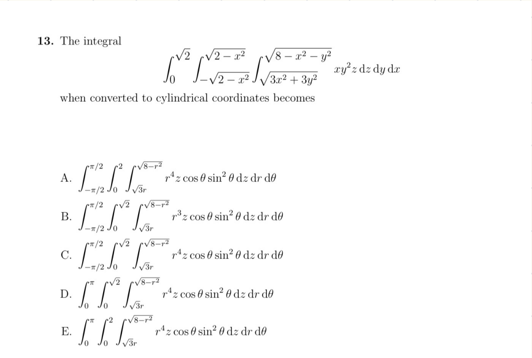
E. $\int_0^\pi\int_0^9\int_0^r 8r^2 dz\,dr\,d\theta$



$\int_0^\pi \int_0^3 \int_0^r 8(r^3)dz\,dr\,d\theta$

θ r

· S19FE#13



13. The integral

$$\int_0^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{\sqrt{3x^2+3y^2}}^{\sqrt{8-x^2-y^2}} xy^2z\,dz\,dy\,dx$$

when converted to cylindrical coordinates becomes

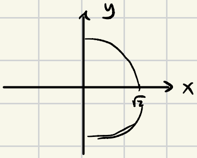
A. $\int_{-\pi/2}^{\pi/2}\int_0^2\int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^4 z\cos\theta\sin^2\theta\,dz\,dr\,d\theta$

B. $\int_{-\pi/2}^{\pi/2}\int_0^{\sqrt{2}}\int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^3 z\cos\theta\sin^2\theta\,dz\,dr\,d\theta$

C. $\int_{-\pi/2}^{\pi/2}\int_0^{\sqrt{2}}\int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^4 z\cos\theta\sin^2\theta\,dz\,dr\,d\theta$

D. $\int_0^{\pi}\int_0^{\sqrt{2}}\int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^4 z\cos\theta\sin^2\theta\,dz\,dr\,d\theta$

E. $\int_0^{\pi}\int_0^2\int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^4 z\cos\theta\sin^2\theta\,dz\,dr\,d\theta$



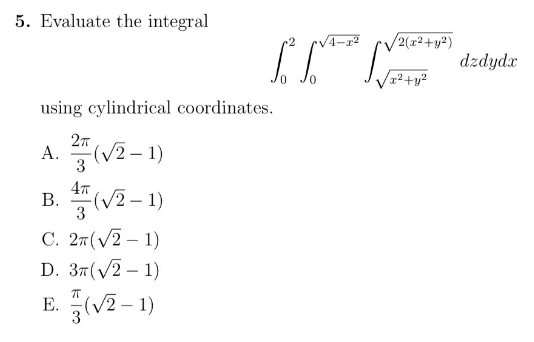
$x = r\cos\theta$

$y = r\sin\theta$

$y^2 = r^2\sin^2\theta$

$\int_0^{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^4\sin^2\theta\cos\theta\, z\; dz\,d\theta\,dr$

- F19E2#5

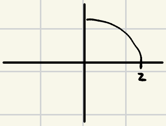


5. Evaluate the integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2(x^2+y^2)}} dz dy dx$$

using cylindrical coordinates.

A. $\frac{2\pi}{3}(\sqrt{2}-1)$
B. $\frac{4\pi}{3}(\sqrt{2}-1)$
C. $2\pi(\sqrt{2}-1)$
D. $3\pi(\sqrt{2}-1)$
E. $\frac{\pi}{3}(\sqrt{2}-1)$



$$\int_0^{\pi/2}\int_0^2\int_r^{r\sqrt{2}} r\,dz dr d\theta$$

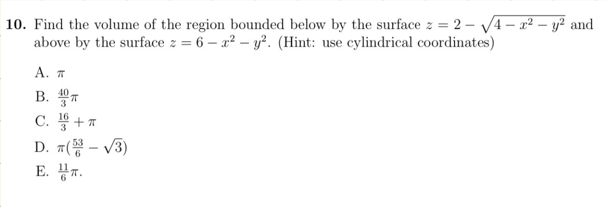
$$r(r\sqrt{2}-r) = r^2\sqrt{2}-r^2$$

$$\int_0^2 r^2\sqrt{2}-r^2\,dr = \frac{\sqrt{2}}{3}r^3-r^3\Big|_0^2 = \frac{8\sqrt{2}}{3}-8$$

$$\int_0^{\pi/2} \frac{8\sqrt{2}}{3}-8\,d\theta = \frac{8\pi\sqrt{2}}{6}-\frac{8\pi}{6} = \frac{8\pi}{6}(\sqrt{2}-1)$$

$$= \frac{4\pi}{3}(\sqrt{2}-1)$$

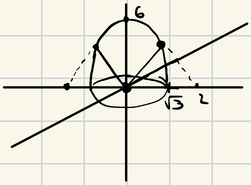
- F19FE #10



10. Find the volume of the region bounded below by the surface $z = 2-\sqrt{4-x^2-y^2}$ and above by the surface $z = 6-x^2-y^2$. (Hint: use cylindrical coordinates)

A. π
B. $\frac{40}{3}\pi$
C. $\frac{16}{3}+\pi$
D. $\pi(\frac{53}{6}-\sqrt{3})$
E. $\frac{11}{6}\pi$.

1) Try to create a sketch
2) Find intersection point to find r



$$2-\sqrt{4-x^2-y^2} \le z \le 6-x^2-y^2$$

$$2-\sqrt{4-r^2} \le z \le 6-r^2$$

$$6-r^2 = 2-\sqrt{4-r^2}$$
$$4-r^2 = -\sqrt{4-r^2}$$
$$(-4+r^2)^2 = 4-r^2$$
$$16-8r^2+r^4 = 4-r^2$$
$$r^4-7r^2+12 = 0$$
$$(r^2-4)(r^2-3)=0$$
$$r=\pm 2 \quad r=\pm\sqrt{3}$$

$$(-z+2)^2 = 4-x^2-y^2$$
$$z^2-2z+4+x^2+y^2 = 4$$
$$z^2-2z+x^2+y^2 = 0$$
$$z(z-2) = 0$$
$$\downarrow$$

$$\longrightarrow \quad 0 \le r \le 2$$

$$\int_0^{2\pi}\int_0^2\int_{2-\sqrt{4-r^2}}^{6-r^2} r\,dz dr d\theta$$

$$r(6-r^2-2+\sqrt{4-r^2})$$

$$\int_0^2 r(4-r^2+\sqrt{4-r^2})dr$$

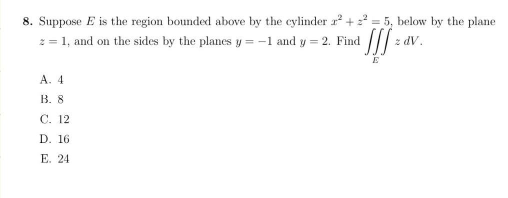
$u = 4-r^2$
$du = -2r$

$$-\frac{1}{2}\int_4^0 (u+\sqrt{u}) = -\frac{1}{2}\left(\frac{1}{2}u^2+\frac{2}{3}u^{3/2}\right)\Big|_4^0$$

$$\frac{1}{2}\left(\frac{16}{2}+\frac{16}{3}\right) = 4+\frac{16}{6} = \frac{40}{6}$$

$$\int_0^{2\pi}\frac{40}{6} = \frac{80\pi}{6} = \frac{40\pi}{3}$$

· S18FE#8



8. Suppose E is the region bounded above by the cylinder $x^2 + z^2 = 5$, below by the plane $z = 1$, and on the sides by the planes $y = -1$ and $y = 2$. Find $\iiint_E z\, dV$.

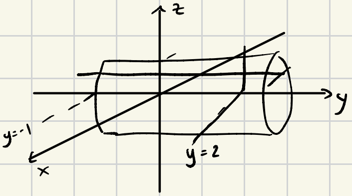
A. 4

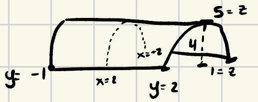
B. 8

C. 12

D. 16

E. 24





$x^2 = 4$

$x = \pm 2$

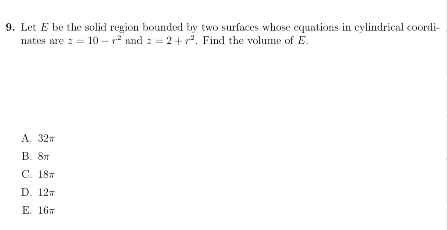
$$\int_{-1}^{2}\int_{-2}^{2}\int_{1}^{\sqrt{5-x^2}} z\, dz\,dx\,dy = \iint \tfrac{1}{2}z^2\Big|_1^{\sqrt{5-x^2}} = \tfrac{1}{2}(5-x^2-1) = \tfrac{1}{2}(4-x^2)$$

$$\tfrac{1}{2}\int_{-2}^{2} 4-x^2 = \tfrac{1}{2}\left(4x - \tfrac{1}{3}x^3\right)\Big|_{-2}^{2} = \tfrac{1}{2}\left[\left(8-\tfrac{8}{3}\right)-\left(-8+\tfrac{8}{3}\right)\right]$$

$$= 8 - \tfrac{16}{6} = \tfrac{32}{6}$$

$$\int_{-1}^{2} \tfrac{32}{6} = 16$$

· F18 E2#9



9. Let E be the solid region bounded by two surfaces whose equations in cylindrical coordinates are $z = 10 - r^2$ and $z = 2 + r^2$. Find the volume of E.

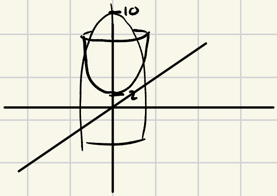
A. 32π

B. 8π

C. 18π

D. 12π

E. 16π



$10 - r^2 = 2 + r^2$

$8 = 2r^2$

$4 = r^2$

$r = \pm 2$

$$\int_0^{2\pi}\int_0^{2}\int_{2+r^2}^{10-r^2} r\, dz\,dr\,d\theta$$

$$r(10-2-r^2-r^2) = 8r - r^3$$

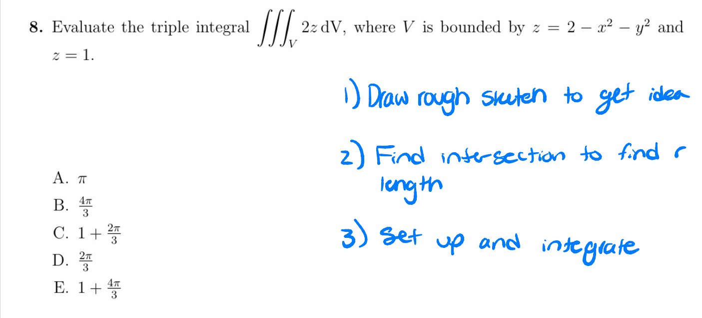
$$\int_0^{2\pi}\int_0^{2} 8r - 2r^3$$

$$= \int_0^{2\pi} 4r^2 - \tfrac{2}{4}r^4\Big|_0^2\, d\theta$$

$$= \int_0^{2\pi} 16 - \tfrac{32}{4}\, d\theta = 8\theta\Big|_0^{2\pi}$$

$$= 16\pi$$

· F18E2#8



8. Evaluate the triple integral $\iiint_V 2z\,\mathrm{dV}$, where V is bounded by $z = 2 - x^2 - y^2$ and $z = 1$.

A. π
B. $\frac{4\pi}{3}$
C. $1 + \frac{2\pi}{3}$
D. $\frac{2\pi}{3}$
E. $1 + \frac{4\pi}{3}$

1) Draw rough sketch to get idea

2) Find intersection to find r length

3) Set up and integrate

$1 = 2 - r^2$

$1 = r^2$

$\pm = r$

$$\int_0^{2\pi}\int_0^1\int_1^{2-r^2} 2zr\,dz\,dr\,d\theta$$

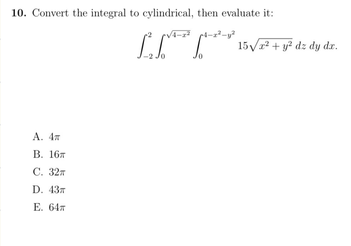
$$\iint z^2 r\Big|_1^{2-r^2} dr\,d\theta = r(4-4r^2+r^4-1)\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 3r - 4r^3 + r^5\,dr\,d\theta$$

$$\frac{3}{2}r^2 - r^4 + \frac{1}{6}r^6\Big|_0^1 = \left(\frac{3}{2} - 1 + \frac{1}{6}\right)d\theta = \frac{9}{6} - \frac{6}{6} + \frac{1}{6} = \frac{4}{6}$$

$$\int_0^{2\pi}\frac{4}{6}\,d\theta = \frac{8\pi}{6} = \frac{4\pi}{3}$$

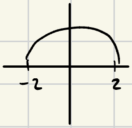
· F18FE#10



10. Convert the integral to cylindrical, then evaluate it:

$$\int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_0^{4-x^2-y^2} 15\sqrt{x^2+y^2}\,dz\,dy\,dx.$$

A. 4π
B. 16π
C. 32π
D. 43π
E. 64π



$$\int_{0}^{\pi}\int_0^2\int_0^{4-r^2} 15r^2\,dz\,dr\,d\theta$$

θ r

$$\iint 15r^2(4-r^2)\,dr\,d\theta = 60r^2 - 15r^4$$

$$\int_0^2 60r^2 - 15r^4\,dr = 20r^3 - 3r^5\Big|_0^2$$

$$= 160 - 3(32) = 160 - 96 = 64$$

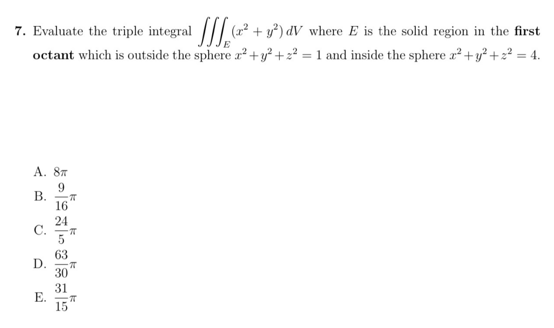
$$\int_0^{\pi} 64 = 64\pi$$

$$\begin{array}{r} 32 \\ \times\ 3 \\ \hline 96 \end{array}$$

$$\begin{array}{r} {}^{15} \\ 0\ 5\ {}^{10} \\ \cancel{160} \\ -\ 96 \\ \hline 64 \end{array}$$

Spherical Coordinates

· S19E2 #7



7. Evaluate the triple integral $\iiint_E (x^2+y^2)\,dV$ where E is the solid region in the **first octant** which is outside the sphere $x^2+y^2+z^2=1$ and inside the sphere $x^2+y^2+z^2=4$.

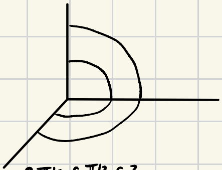
A. 8π
B. $\frac{9}{16}\pi$
C. $\frac{24}{5}\pi$
D. $\frac{63}{30}\pi$
E. $\frac{31}{15}\pi$

$x=\rho\sin\phi\cos\theta$

$y=\rho\sin\phi\sin\theta$

$\rho^2=x^2+y^2+z^2$

$dV=\rho^2\sin\phi$



$0\le\phi\le\pi/2$

$0\le\theta\le\pi/2$

$0\le\rho\le 2$

$x^2+y^2=\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta$

$=\rho^2(\sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta)$

$=\rho^2\sin^2\phi$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^4\sin^3\phi\;d\rho\,d\phi\,d\theta$$

$$\tfrac{1}{5}\rho^5\sin^3\phi\Big|_1^2=\tfrac{1}{5}\sin^3\phi\,(32-1)=\tfrac{31}{5}\sin^3\phi$$

$$\tfrac{31}{5}\int_0^{\pi/2}\sin\phi\,(1-\cos^2\phi)$$

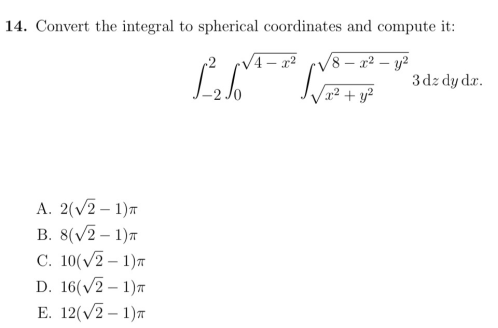
$u=\cos\phi$

$du=-\sin\phi$

$$-\tfrac{31}{5}\int_1^0 1-u^2\,du=-\tfrac{31}{5}\left(u-\tfrac{1}{3}u^3\right)\Big|_1^0=-\tfrac{31}{5}\left(-1+\tfrac{1}{3}\right)=-\tfrac{31}{5}\cdot\tfrac{-2}{3}=\tfrac{62}{15}$$

$$\int_0^{\pi/2}\left(\tfrac{31}{5}\cdot\tfrac{2}{3}\right)d\theta=\frac{31\pi}{15}$$

· S19FE#14



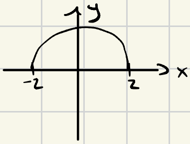
14. Convert the integral to spherical coordinates and compute it:

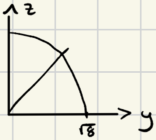
$$\int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 3\,dz\,dy\,dx.$$

A. $2(\sqrt{2}-1)\pi$
B. $8(\sqrt{2}-1)\pi$
C. $10(\sqrt{2}-1)\pi$
D. $16(\sqrt{2}-1)\pi$
E. $12(\sqrt{2}-1)\pi$

$x=\rho\cos\theta\sin\phi$

$y=\rho\sin\phi\sin\theta$





$0\le\phi\le\pi/4$

$0\le\theta\le\pi$

$0\le\rho\le\sqrt{8}$

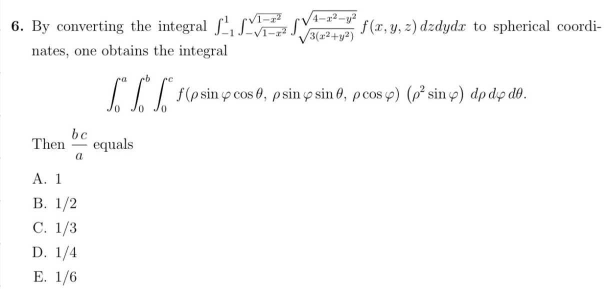
$$\int_0^{\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} 3\rho^2\sin\phi\;d\rho\,d\phi\,d\theta$$

$$\rho^3\sin\phi\Big|_0^{\sqrt{8}}=8\sqrt{8}\sin\phi$$

$$-8\sqrt{8}\cos\phi\Big|_0^{\pi/4}=-8\sqrt{8}\left(\tfrac{1}{\sqrt{2}}-1\right)=-\tfrac{8\cdot2\sqrt{2}}{\sqrt{2}}+8\sqrt{8}=-16+16\sqrt{2}$$

$$\int_0^{\pi}-16+16\sqrt{2}=16\pi(\sqrt{2}-1)$$

· F19E2#6

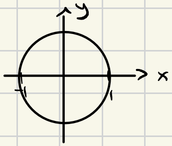


6. By converting the integral $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{3(x^2+y^2)}}^{\sqrt{4-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$ to spherical coordinates, one obtains the integral

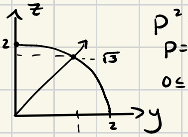
$$\int_0^a\int_0^b\int_0^c f(\rho\sin\varphi\cos\theta,\,\rho\sin\varphi\sin\theta,\,\rho\cos\varphi)\,(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta.$$

Then $\frac{b\,c}{a}$ equals

- A. 1
- B. 1/2
- C. 1/3
- D. 1/4
- E. 1/6



$0 \le \theta \le 2\pi$



$\rho^2 = 4$

$\rho = \pm 2$

$0 \le \rho \le 2$

$z = \rho\cos\phi$

$\sqrt{3} = 2\cos\phi$

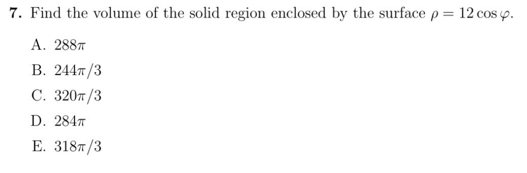
$\frac{\sqrt{3}}{2} = \cos\phi$

$\phi = \frac{\pi}{6}$

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\frac{bc}{a} = \frac{\frac{2\pi}{6}}{2\pi} = \frac{2\pi}{12\pi} = \frac{1}{6}$$

· F19E2#7



7. Find the volume of the solid region enclosed by the surface $\rho = 12\cos\varphi$.

- A. 288π
- B. $244\pi/3$
- C. $320\pi/3$
- D. 284π
- E. $318\pi/3$

$$\begin{array}{r} 144 \\ \times\ 12 \\ \hline 288 \\ 1440 \\ \hline 1728 \end{array}$$

$z = \rho\cos\phi$

$\rho^2 = x^2+y^2+z^2$

$x^2+y^2+z^2 = 12z$

$x^2+y^2+z^2-12z+\underline{36} = \underline{36}$

$x^2+y^2+(z-6)^2 = 36$

$0 \le \theta \le 2\pi$

$0 \le \rho \le 12\cos\phi$

$0 \le \phi \le \pi/2$

$\rho = 12\cos\phi$

$0 = 12\cos\phi$

$\pi/2 = \phi$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{12\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\frac{1}{3}\rho^3\Big|_0^{12\cos\phi} = \frac{1728}{3}\cos^3\phi$$

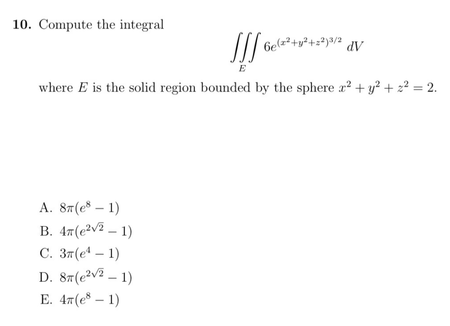
$$\frac{1728}{3}\int_0^{\pi/2}\cos^3\phi\sin\phi = -\frac{1728}{3}\int_1^0 u^3du = -\frac{1728}{3}\left(\frac{1}{4}\right) = \frac{1728}{12}$$

$u = \cos\phi$

$du = -\sin\phi$

$$\int_0^{2\pi} 144\,d\theta = 288\pi$$

· F18E2#10



10. Compute the integral

$$\iiint_E 6e^{(x^2+y^2+z^2)^{3/2}}\,dV$$

where E is the solid region bounded by the sphere $x^2+y^2+z^2=2$.

- A. $8\pi(e^8-1)$
- B. $4\pi(e^{2\sqrt{2}}-1)$
- C. $3\pi(e^4-1)$
- D. $8\pi(e^{2\sqrt{2}}-1)$
- E. $4\pi(e^8-1)$

$\rho^2 = 2$

$0 \le \rho \le \sqrt{2}$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{2}} 6e^{(\rho^2)^{3/2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\iiint 6e^{\rho^3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$u = \rho^3$

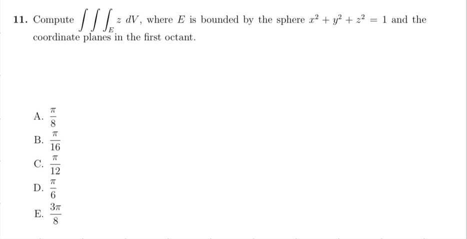
$du = 3\rho^2$

$$= \frac{1}{3}\int_0^{\sqrt{2}} 6e^u\sin\phi = 2\sin\phi\,(e^{\sqrt{2}}-1)$$

$$2(e^{\sqrt{2}}-1)\int_0^{\pi}\sin\phi = -2(e^{\sqrt{2}}-1)\cos\phi\Big|_0^{\pi} = -2(e^{\sqrt{2}}-1)(-1-1) = 4(e^{\sqrt{2}}-1)$$

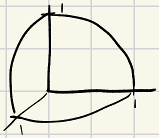
$$\int_0^{2\pi} 4(e^{\sqrt{2}}-1) = 8\pi(e^{\sqrt{2}}-1)$$

· F18 FE 11



11. Compute $\iiint_E z\,dV$, where E is bounded by the sphere $x^2+y^2+z^2=1$ and the coordinate planes in the first octant.

A. $\frac{\pi}{8}$
B. $\frac{\pi}{16}$
C. $\frac{\pi}{12}$
D. $\frac{\pi}{6}$
E. $\frac{3\pi}{8}$



$0 \le \phi \le \pi/2$
$0 \le \theta \le \pi/2$
$0 \le \rho \le 1$

$$\underset{\theta}{\int_0^{\pi/2}}\underset{\phi}{\int_0^{\pi/2}}\underset{\rho}{\int_0^1} \rho\cos\phi\,\rho^2\sin\phi\;d\rho\,d\phi\,d\theta$$

$$\cos\phi\sin\phi\left(\frac{1}{4}\rho^3\right)\Big|_0^1 = \frac{1}{4}\cos\phi\sin\phi$$

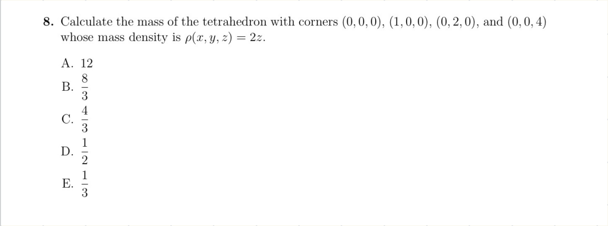
$$\frac{1}{4}\int_0^{\pi/2}\cos\phi\,\sin\phi \qquad \begin{array}{l}u=\sin\phi\\ du=\cos\phi\end{array}$$

$$\frac{1}{4}\int_0^1 u\,du = \frac{1}{4}\left(\frac{1}{2}u^2\right)\Big|_0^1 = \frac{1}{8}$$

$$\int_0^{\pi/2}\frac{1}{8} = \frac{\pi}{16}$$

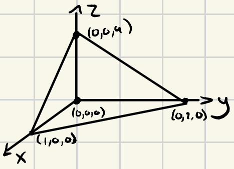
16.6 Integrals for Mass Calculation

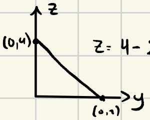
· F19E2#8



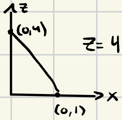
8. Calculate the mass of the tetrahedron with corners (0, 0, 0), (1, 0, 0), (0, 2, 0), and (0, 0, 4) whose mass density is $\rho(x,y,z)=2z$.

A. 12
B. $\frac{8}{3}$
C. $\frac{4}{3}$
D. $\frac{1}{2}$
E. $\frac{1}{3}$

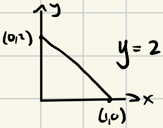




$z = 4 - 2y$



$z = 4 - 4x$

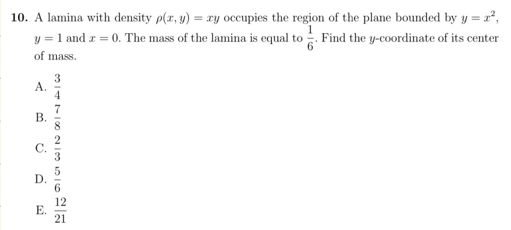


$y = 2 - 2x$

$$\underset{x}{\int_0^1}\underset{y}{\int_0^{2-2x}}\int_0^{4-2y-4x} 2z\,dz\,dy\,dx = \iint (4-2y-4x)^2\,dy\,dx$$

$$\int_0^1\int_0^{2-2x}\left(16x^2-32x-16y+16xy+4y^2+16\right)dy = \frac{8}{3}$$

· S18FE#10



10. A lamina with density $\rho(x,y) = xy$ occupies the region of the plane bounded by $y = x^2$, $y = 1$ and $x = 0$. The mass of the lamina is equal to $\frac{1}{6}$. Find the y-coordinate of its center of mass.

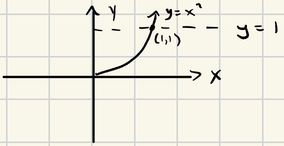
A. $\frac{3}{4}$

B. $\frac{7}{8}$

C. $\frac{2}{3}$

D. $\frac{5}{6}$

E. $\frac{12}{21}$



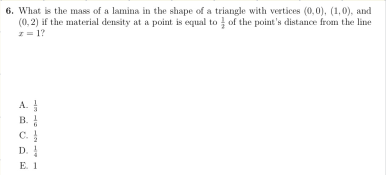
$m = \frac{1}{6}$

$y_{cm} = 6 \int_0^1 \int_{x^2}^1 xy^2 \, dy dx$

$6x \int_{x^2}^1 \frac{1}{3} y^3 = 6x\left(\frac{1}{3} - \frac{x^6}{3}\right) = (2x - 2x^7)$

$\int_0^1 2x - 2x^7 = x^2 - \frac{2}{8}x^8 \Big|_0^1 = 1 - \frac{2}{8} = \frac{6}{8} = \frac{3}{4}$

· F18E2 #6



6. What is the mass of a lamina in the shape of a triangle with vertices (0, 0), (1, 0), and (0, 2) if the material density at a point is equal to $\frac{1}{2}$ of the point's distance from the line $x = 1$?

A. $\frac{1}{3}$

B. $\frac{1}{6}$

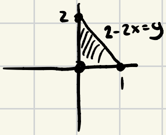
C. $\frac{1}{2}$

D. $\frac{1}{4}$

E. 1

$D = \frac{m}{V}$

$m = \iint \rho(x,y) \, dx \, dy$



$d = \sqrt{(x-1)^2}$

$d = |(x-1)|$

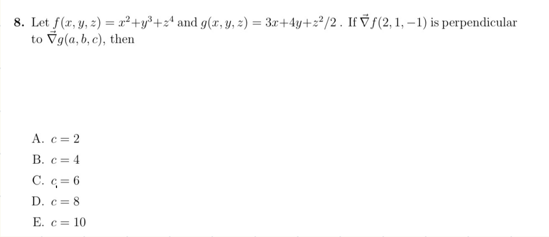
$\rho = \frac{1}{2}(x-1)$

$= \frac{x}{2} - \frac{1}{2}$

$\text{distance} = \sqrt{(x_0-x)^2 + (y_0-y)^2}$

17.1 Vector Fields

- S19E2#8



8. Let $f(x,y,z) = x^2+y^3+z^4$ and $g(x,y,z) = 3x+4y+z^2/2$. If $\vec{\nabla} f(2,1,-1)$ is perpendicular to $\vec{\nabla} g(a,b,c)$, then

A. $c = 2$
B. $c = 4$
C. $c = 6$
D. $c = 8$
E. $c = 10$

$\nabla g = \langle 3, 4, z\rangle$

$\nabla f = \langle 2x, 3y^2, 4z^3\rangle$

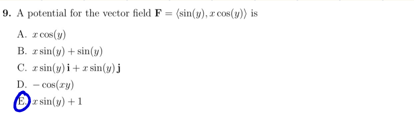
$\nabla f_{(2,1,-1)} = \langle 4, 3, -4\rangle$

$\nabla f_{(2,1,-1)} \cdot \nabla g = 0 = \langle 3,4,z\rangle \cdot \langle 4,3,-4\rangle = 12+12-4z$

$= 24-4z$

$z=6$

- F19E2#9



9. A potential for the vector field $\mathbf{F} = \langle \sin(y), x\cos(y)\rangle$ is

A. $x\cos(y)$
B. $x\sin(y)+\sin(y)$
C. $x\sin(y)\,\mathbf{i} + x\sin(y)\,\mathbf{j}$
D. $-\cos(xy)$
[✓ E.] $x\sin(y)+1$

$P = \sin y \qquad Q = x\cos y$

$P_y = \cos y \qquad Q_x = \cos y \qquad P = Q \quad \therefore$ conservative

$\frac{df}{dy} = x\cos y\, dy$

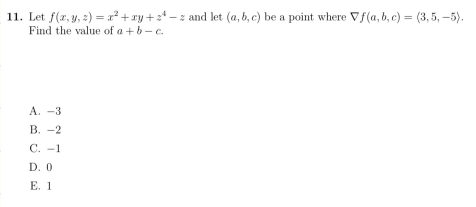
$f(x,y) = \int \sin y\, dx = X\sin y + g(y)$

$x\sin y + g'(y) = x\cos y dy$

$g'(y) = 0$

$f(x,y) = x\sin y + C \longrightarrow X\sin y + 1$

- F18E2#11



11. Let $f(x,y,z) = x^2+xy+z^4-z$ and let (a,b,c) be a point where $\nabla f(a,b,c) = \langle 3,5,-5\rangle$. Find the value of $a+b-c$.

A. -3
B. -2
C. -1
D. 0
E. 1

$\nabla f = \langle 2x+y, x, 4z^3-1\rangle$

$\nabla f_{(a,b,c)} = \langle 2a+b, a, 4c^3-1\rangle = \langle 3,5,-5\rangle$

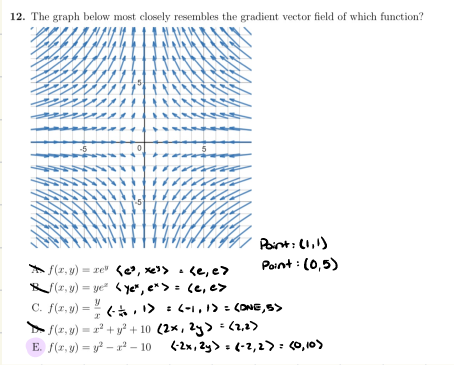
$2a+b=3 \qquad a=5 \qquad 4c^3-1=-5 \qquad a+b-c$

$10+b=3 \qquad 4c^3=-4 \qquad 5-7+1 = -1$

$b=-7 \qquad c^3=-1$

$c=-1$

- F18E2# 12



12. The graph below most closely resembles the gradient vector field of which function?

Point: (1,1)

Point: (0,5)

~~A.~~ $f(x,y) = xe^y$ $\langle e^y, xe^y\rangle = \langle e, e\rangle$

~~B.~~ $f(x,y) = ye^x$ $\langle ye^x, e^x\rangle = \langle e, e\rangle$

C. $f(x,y) = \frac{y}{x}$ $\langle -\frac{1}{x^2}, 1\rangle = \langle -1, 1\rangle = \langle \text{DNE}, 5\rangle$

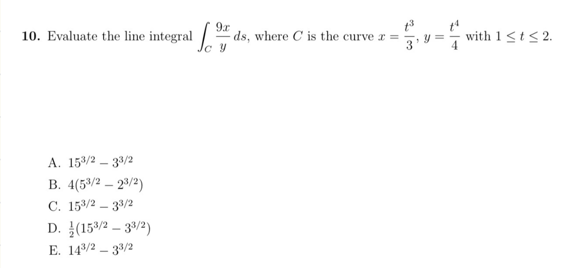
~~D.~~ $f(x,y) = x^2+y^2+10$ $\langle 2x, 2y\rangle = \langle 2,2\rangle$

[✓ E.] $f(x,y) = y^2-x^2-10$ $\langle -2x, 2y\rangle = \langle -2, 2\rangle = \langle 0, 10\rangle$

17.2 Line Integrals of Functions and Vector Fields

Scalar: For regular lines $\int_C F\,ds = \int_C F|r'(t)|\,dt$

· S19E2#10



10. Evaluate the line integral $\int_C \frac{9x}{y}\,ds$, where C is the curve $x = \frac{t^3}{3}$, $y = \frac{t^4}{4}$ with $1 \le t \le 2$.

A. $15^{3/2} - 3^{3/2}$
B. $4(5^{3/2} - 2^{3/2})$
C. $15^{3/2} - 3^{3/2}$
D. $\frac{1}{2}(15^{3/2} - 3^{3/2})$
E. $14^{3/2} - 3^{3/2}$

$\int_1^2 \frac{3t^3}{t^4/4}$

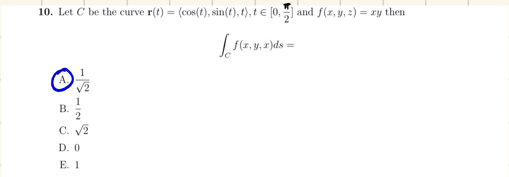
$r'(t) = \langle t^2, t^3 \rangle$ $\quad |r'(t)| = \sqrt{t^4 + t^6} = \sqrt{t^4(1+t^2)} = t^2\sqrt{1+t^2}$

$\int_1^2 \frac{12}{t} \cdot t^2\sqrt{1+t^2} = \int_1^2 12t\sqrt{1+t^2} = 6\int_2^5 u^{1/2} = 6\left(\frac{2}{3}u^{3/2}\right)\Big|_2^5$

$u = 1+t^2$
$du = 2t$

$= 4(5^{3/2} - 2^{3/2})$

· F19E2#10



10. Let C be the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle, t \in [0, \frac{\pi}{2}]$ and $f(x,y,z) = xy$ then

$$\int_C f(x,y,x)ds =$$

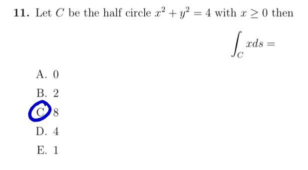
[✓ A.] $\frac{1}{\sqrt{2}}$
B. $\frac{1}{2}$
C. $\sqrt{2}$
D. 0
E. 1

$\int_0^{\pi/2} \cos t \sin t = \sqrt{2}\int_0^{\pi/2} \cos t \sin t$ $\quad u = \sin t$, $du = \cos t$

$r'(t) = \langle -\sin t, \cos t, 1 \rangle$ $\quad \sqrt{2}\int_0^1 \frac{1}{2}u^2 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$|r'(t)| = \sqrt{2}$

· F19E2#11



11. Let C be the half circle $x^2 + y^2 = 4$ with $x \ge 0$ then

$$\int_C x\,ds =$$

A. 0
B. 2
[✓ C.] 8
D. 4
E. 1

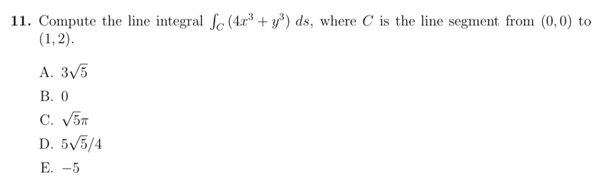
$y = \sqrt{4 - x^2}$

$r(t) = \langle t, \sqrt{4-t^2} \rangle$

$r'(t) = \langle 1,$

$|r'(t)| =$

· F19FE#11



11. Compute the line integral $\int_C (4x^3 + y^3)\,ds$, where C is the line segment from $(0,0)$ to $(1,2)$.

A. $3\sqrt{5}$
B. 0
C. $\sqrt{5}\pi$
D. $5\sqrt{5}/4$
E. -5

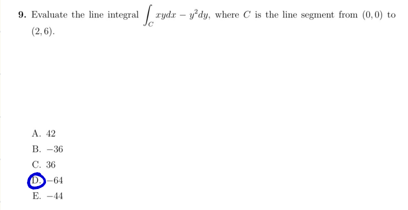
$r(t) = \langle 0,0 \rangle + t\langle 1,2 \rangle = \langle t, 2t \rangle$ $\quad 0 \le t \le 1$

$r'(t) = \langle 1, 2 \rangle$ $\quad |r'(t)| = \sqrt{5}$

$\int_0^1 [4t^3 + (2t)^3]\sqrt{5} = \sqrt{5}\int_0^1 12t^3 = 3\sqrt{5}\,t^4\Big|_0^1 = 3\sqrt{5}$

Vectors: $\int_a^b F \cdot r' dt$

· S19E2#9



9. Evaluate the line integral $\int_C xydx - y^2dy$, where C is the line segment from $(0,0)$ to $(2,6)$.

A. 42
B. −36
C. 36
[✓ D.] −64
E. −44

$r(t) = \langle 0,0\rangle + t\langle 2,6\rangle = \langle 2t, 6t\rangle \qquad 0 \le t \le 1$

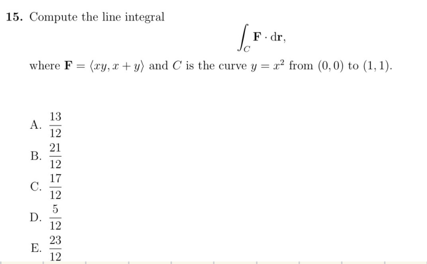
$x = 2t \quad dx = 2$

$y = 6t \quad dy = 6$

$\int_0^1 12t^2(2) - 36t^2(6)\, dt$

$\int_0^1 24t^2 - 216t^2\, dt = -64$

· S19FE #15



15. Compute the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where $\mathbf{F} = \langle xy, x+y\rangle$ and C is the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

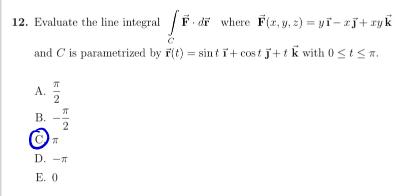
A. $\frac{13}{12}$
B. $\frac{21}{12}$
C. $\frac{17}{12}$
D. $\frac{5}{12}$
E. $\frac{23}{12}$

$r(t) = \langle t, t^2\rangle \quad r'(t) = \langle 1, 2t\rangle$

$\int_0^1 \langle t^3, t+t^2\rangle \cdot \langle 1, 2t\rangle = \int_0^1 t^3 + 2t^2 + 2t^3 = 3t^3 + 2t^2 = \frac{3}{4}t^4 + \frac{2}{3}t^2\Big|_0^1$

$= \frac{9}{12} + \frac{8}{12} = \frac{17}{12}$

· S18 FE #12



12. Evaluate the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}}(x,y,z) = y\,\vec{\mathbf{i}} - x\,\vec{\mathbf{j}} + xy\,\vec{\mathbf{k}}$ and C is parametrized by $\vec{\mathbf{r}}(t) = \sin t\,\vec{\mathbf{i}} + \cos t\,\vec{\mathbf{j}} + t\,\vec{\mathbf{k}}$ with $0 \le t \le \pi$.

A. $\frac{\pi}{2}$
B. $-\frac{\pi}{2}$
[✓ C.] π
D. $-\pi$
E. 0

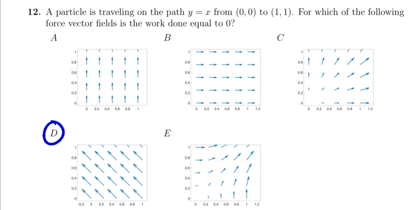
$r'(t) = \langle \cos t, -\sin t, 1\rangle$

$\int_0^\pi \langle \cos t, -\sin t, 1\rangle \cdot \langle \cos t, -\sin t, -\sin t\cos t\rangle$

$\int_0^\pi \cos^2 t + \sin^2 t - \sin t\cos t = \int_0^\pi 1 - \int_0^\pi \sin t\cos t \qquad u = \sin t \quad du = \cos t$

$- \int_0^\pi 1 - \int_0^0 u\,du = \pi$

· F18FE#12



12. A particle is traveling on the path $y = x$ from $(0,0)$ to $(1,1)$. For which of the following force vector fields is the work done equal to 0?

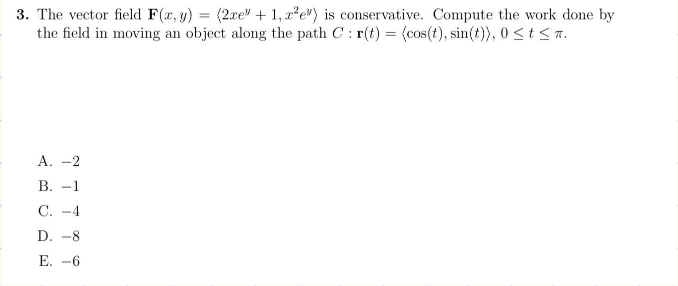
$W = \int_a^b F \cdot r'(t)dt$

$r(t) = \langle t, t\rangle \qquad 0 \le t \le 1$

$r'(t) = \langle 1, 1\rangle$

17.3 Conservative Vector Fields & Fundemental Theorem of Line Integrals

- S19 FE #3



3. The vector field $\mathbf{F}(x,y) = \langle 2xe^y + 1, x^2e^y \rangle$ is conservative. Compute the work done by the field in moving an object along the path $C : \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$, $0 \le t \le \pi$.

A. -2
B. -1
C. -4
D. -8
E. -6

$r'(t) = \langle -\sin t, \cos t \rangle$

$F = \langle 2\cos t\, e^{\sin t} + 1, \cos^2 t\, e^{\sin t} \rangle$

$\int_0^\pi F \cdot r'\, dt = \int_0^\pi -2\sin t\cos t\, e^{\sin t} - \sin t + \cos^3 t\, e^{\sin t}$

$\int_0^\pi -2\sin t\cos t\, e^{\sin t} - \int_0^\pi \sin t + \int_0^\pi \cos^3 t\, e^{\sin t}$

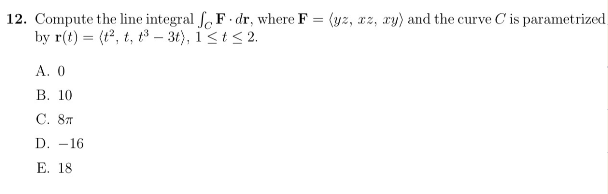
$u = \sin t \quad du = \cos t$

$0 \quad + \cos t \,\Big|_0^\pi + \int_0^\pi \cos t\,(\cos^2 t)\, e^{\sin t}$

$-2 + \left[2\left(-e^{\sin(t)} + e^{\sin(t)} + \sin t\right) + e^{\sin(t)}\cos^2(t)\right]\Big|_0^\pi$

-2

- F19 FE #12



12. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle yz, xz, xy \rangle$ and the curve C is parametrized by $\mathbf{r}(t) = \langle t^2, t, t^3 - 3t \rangle$, $1 \le t \le 2$.

A. 0
B. 10
C. 8π
D. -16
E. 18

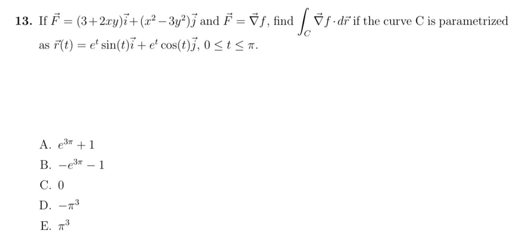
$r'(t) = \langle 2t, 1, 3t^2 - 3 \rangle$

$F = \langle t^4 - 3t^2, t^5 - 3t^3, t^3 \rangle$

$\int_1^2 2t(t^4 - 3t^2) + (t^5 - 3t^3) + t^3(3t^2 - 3) = \int_1^2 6t^5 - 12t^3\, dt$

$t^6 - 3t^4 \Big|_1^2 = (64 - 48) - (1 - 3) = 18$

· F18FE#13



13. If $\vec{F} = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j}$ and $\vec{F} = \vec{\nabla} f$, find $\int_C \vec{\nabla} f \cdot d\vec{r}$ if the curve C is parametrized as $\vec{r}(t) = e^t \sin(t)\vec{i} + e^t \cos(t)\vec{j}$, $0 \le t \le \pi$.

A. $e^{3\pi} + 1$
B. $-e^{3\pi} - 1$
C. 0
D. $-\pi^3$
E. π^3

If $f = \nabla f$ then: $\int F \cdot dr = f \Big|_{start}^{end}$

$$\int_c F \cdot dr = f \Big|_{start}^{end}$$

$$\int F = \langle 3x + x^2 y ,\ x^2 y - y^3 \rangle$$

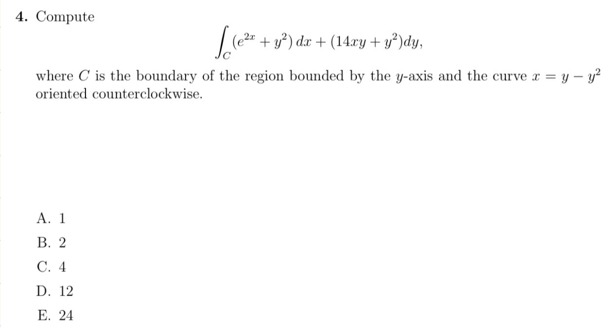
$$r(0) = (0,1)$$
$$r(\pi) = (0, -e^{\pi})$$

$$f = 3x + x^2 y - y^3 \Big|_{(0,1)}^{(0,-e^{\pi})} \quad (0,-e^{\pi}) = -(-e^{\pi})^3 = e^{3\pi} \quad (0,1) = -1$$

$$= e^{3\pi} - (-1) = e^{3\pi} + 1$$

17.4 Green's Theorem

· S19FE#4

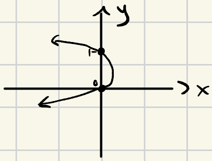


4. Compute
$$\int_C (e^{2x} + y^2)\,dx + (14xy + y^2)dy,$$
where C is the boundary of the region bounded by the y-axis and the curve $x = y - y^2$ oriented counterclockwise.

A. 1
B. 2
C. 4
D. 12
E. 24

$x = y - y^2$
$y = y^2$
$y = 1$

$0 \le x \le y - y^2$
$0 \le y \le 1$



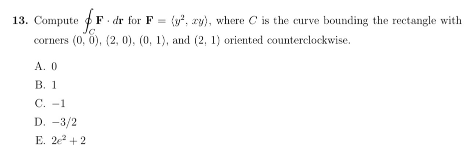
$$\int_c P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$P = e^{2x} + y^2 \qquad Q = 14xy + y^2$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 14y - 2y$$

$$\underset{y}{\int_0^1} \underset{x}{\int_0^{y-y^2}} 12y \, dx dy = \int_0^1 12y^2 - 12y^3$$
$$= 4y^3 - 3y^4 = 1$$

· F19FE #13

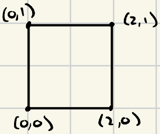


13. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle y^2, xy \rangle$, where C is the curve bounding the rectangle with corners (0, 0), (2, 0), (0, 1), and (2, 1) oriented counterclockwise.

A. 0
B. 1
C. -1
D. $-3/2$
E. $2e^2 + 2$

$$\oint F \cdot dr = \oint_c P dx + Q dx = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

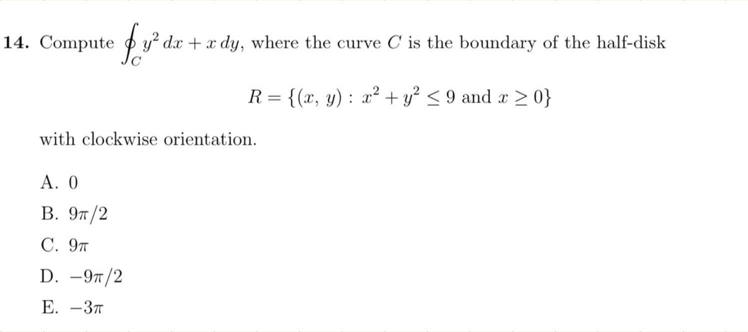
$F = \langle y^2 , xy \rangle \qquad P = y^2 \qquad Q = xy$



$0 \leq x \leq 2$
$0 \leq y \leq 1$

$$\int_0^1 \int_0^2 (y - 2y) \, dx \, dy = 2\int_0^1 (y - 2y) \, dy = 2\left(\tfrac{1}{2}y^2 - y^2\right)\Big|_0^1 = -1$$

· F19FE #14

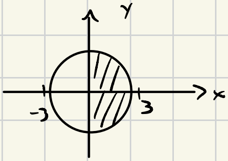


14. Compute $\oint_C y^2 \, dx + x \, dy$, where the curve C is the boundary of the half-disk

$$R = \{(x, y) : x^2 + y^2 \leq 9 \text{ and } x \geq 0\}$$

with clockwise orientation.

A. 0
B. $9\pi/2$
C. 9π
D. $-9\pi/2$
E. -3π



$$\oint P dx + Q dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$P = y^2 \qquad Q = x$

$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = 1$

$-\pi/2 \leq \theta \leq \pi/2$

$0 \leq r \leq 3$

Added since going in cw direction

$$-\iint (1 - 2y) \, dx dy = -\int_{-\pi/2}^{\pi/2} \int_0^3 r(1 - 2r\sin\theta) \, dr d\theta$$

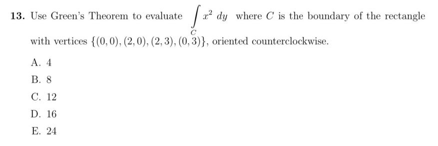
$$= -\left[\int_{-\pi/2}^{\pi/2} \int_0^3 r - \int_{-\pi/2}^{\pi/2} \int_0^3 2r^2 \sin\theta \right] dr d\theta$$

$$= -\left[\int_{-\pi/2}^{\pi/2} \tfrac{1}{2} r^2 \Big|_0^3 - \int_{-\pi/2}^{\pi/2} \tfrac{1}{3} 2r^3 \sin\theta \Big|_0^3 \right]$$

$$= -\left[\tfrac{9}{2}\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}\right) - \int_{-\pi/2}^{\pi/2} 18 \sin\theta \right]$$

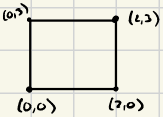
$$= -\frac{9\pi}{2} - 0 = \frac{-9\pi}{2}$$

S18FE#13



13. Use Green's Theorem to evaluate $\int_C x^2\, dy$ where C is the boundary of the rectangle with vertices $\{(0,0),(2,0),(2,3),(0,3)\}$, oriented counterclockwise.

A. 4
B. 8
C. 12
D. 16
E. 24



$$\oint P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$P = 0$

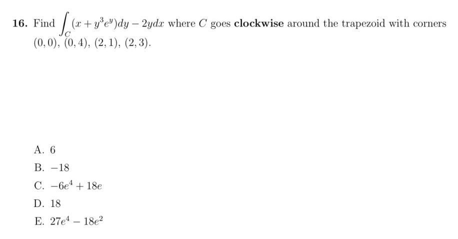
$Q = x^2$

$$\underbrace{\int_0^3}_{y} \underbrace{\int_0^2}_{x} 2x\,dx\,dy = \int_0^3 x^2 \Big|_0^2 dy = 4y \Big|_0^3 = 12$$

$0 \le x \le 2$

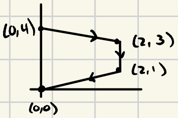
$0 \le y \le 3$

F18FE#16



16. Find $\int_C (x + y^3 e^y)dy - 2y dx$ where C goes **clockwise** around the trapezoid with corners $(0,0)$, $(0,4)$, $(2,1)$, $(2,3)$.

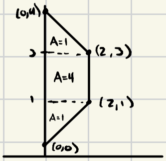
A. 6
B. −18
C. $-6e^4 + 18e$
D. 18
E. $27e^4 - 18e^2$



$$\oint P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

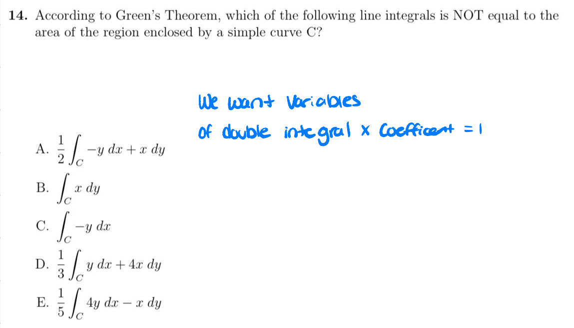
$P = -2y$ $\qquad \frac{\partial P}{\partial y} = -2$

$Q = (x + y^3 e^y)$ $\qquad \frac{\partial Q}{\partial x} = 1$



$$\iint (3)\,dA = 3(6) = 18$$

F18FE#14



14. According to Green's Theorem, which of the following line integrals is NOT equal to the area of the region enclosed by a simple curve C?

A. $\frac{1}{2}\int_C -y\,dx + x\,dy$
B. $\int_C x\,dy$
C. $\int_C -y\,dx$
D. $\frac{1}{3}\int_C y\,dx + 4x\,dy$
E. $\frac{1}{5}\int_C 4y\,dx - x\,dy$

We want variables of double integral x coefficient = 1

$$\int_c F \cdot dr = \int_c P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

A) $P = -y$, $Q = x$ $\qquad \frac{1}{2}\iint (1+1) = \iint 1\,dA$

B) $P = 0$, $Q = x$ $\qquad \iint (1)\,dA$

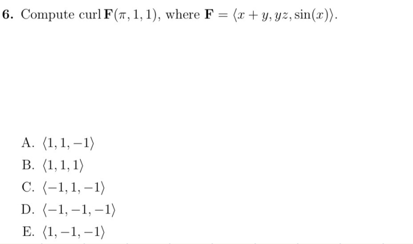
C) $P = -y$, $Q = 0$ $\qquad \iint 1\,dA$

D) $P = y$, $Q = 4x$ $\qquad \frac{1}{3}\iint (4-1) = \iint 1\,dA$

E) $P = 4y$, $Q = -x$ $\qquad \frac{1}{5}\iint (-1-4)\,dA = \iint (-1)\,dA$

17.5 Divergence & Curl

· S19 FE # 6



6. Compute curl $\mathbf{F}(\pi, 1, 1)$, where $\mathbf{F} = \langle x + y, yz, \sin(x)\rangle$.

A. $\langle 1, 1, -1\rangle$
B. $\langle 1, 1, 1\rangle$
C. $\langle -1, 1, -1\rangle$
D. $\langle -1, -1, -1\rangle$
E. $\langle 1, -1, -1\rangle$

$\vec{\nabla} \times F$

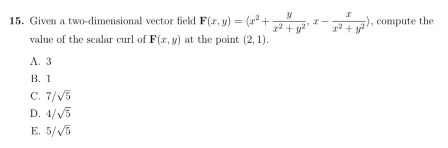
Write out as Variables then plug in and Solve

$$\nabla \times F = \begin{vmatrix} \partial x , & \partial y , & \partial z \\ a & b & c \end{vmatrix} = \langle c_y - b_z, -(c_x - a_z), b_x - a_y \rangle$$

$$= \langle 0 - y, \cos(x) - 0, 0 - 1 \rangle$$

$$= \langle -1, 1, -1 \rangle$$

· F19 FE # 15



15. Given a two-dimensional vector field $\mathbf{F}(x, y) = \langle x^2 + \frac{y}{x^2 + y^2}, x - \frac{x}{x^2 + y^2}\rangle$, compute the value of the scalar curl of $\mathbf{F}(x, y)$ at the point $(2, 1)$.

A. 3
B. 1
C. $7/\sqrt{5}$
D. $4/\sqrt{5}$
E. $5/\sqrt{5}$

$$\text{Curl } F = \langle \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \rangle \vec{K} \longrightarrow$$

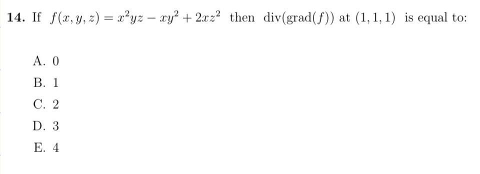
Derived from cross product : $\begin{vmatrix} \partial x & \partial y \\ a & b \end{vmatrix} = \langle b_x - a_y \rangle$

$\vec{K}$ determines direction (fixes $P_y - Q_x$ if flipped)

$$= \langle \underbrace{(1 - \frac{-x^2 + y^2}{(x^2+y^2)^2})}_{bx} \underbrace{-}_{-} \underbrace{\frac{x^2 - y^2}{(x^2+y^2)^2}}_{ax} \rangle = 1$$

$$|\text{curl } F| = 1$$

· S18 FE # 14



14. If $f(x, y, z) = x^2yz - xy^2 + 2xz^2$ then div(grad(f)) at $(1, 1, 1)$ is equal to:

A. 0
B. 1
C. 2
D. 3
E. 4

$$\text{div } F = \nabla \cdot F$$

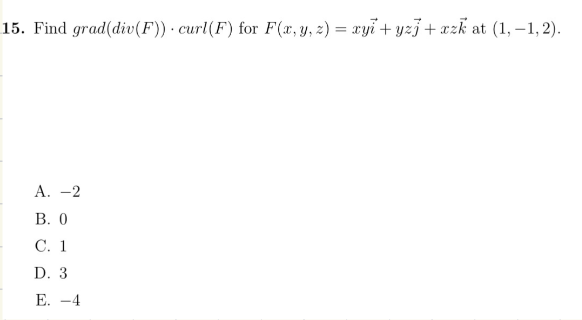
$$\text{div } \nabla_{\nabla} F = \nabla_{\nabla} \cdot F = \langle \partial xx, \partial yy, \partial zz \rangle \cdot \langle 1, 1, 1 \rangle$$

$$\nabla = \langle 2xyz - y^2 + 2z^2, x^2z - 2xy, x^2y + 4xz \rangle$$

$$\nabla_{\nabla} = \langle 2yz, -2x, 4x \rangle$$

$$\text{div } \nabla_{\nabla} F = 2yz - 2x + 4x = 2yz + 2x = 4$$

- F18FE #15



15. Find $grad(div(F)) \cdot curl(F)$ for $F(x,y,z) = xy\vec{i} + yz\vec{j} + xz\vec{k}$ at $(1,-1,2)$.

A. -2
B. 0
C. 1
D. 3
E. -4

$F = \langle xy, yz, xz \rangle$

$\text{div } F = \nabla \cdot F = \langle \partial x, \partial y \; \partial z \rangle \cdot \langle xy, yz, xz \rangle = y + z + x$

$\nabla(\text{div } F) = \langle 1, 1, 1 \rangle$

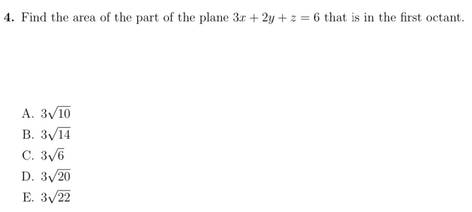
$\text{Curl } F = \begin{vmatrix} \partial x & \partial y & \partial z \\ a & b & c \end{vmatrix} = \langle c_y - b_z, -(c_x - a_z), b_x - a_y \rangle$

$= \langle 0 - y, -(z - 0), 0 - x \rangle = \langle -y, -z, -x \rangle$

$\nabla(\text{div } F) \cdot \text{Curl } F = \langle 1,1,1 \rangle \cdot \langle -y, -z, -x \rangle = -y - z - x = 1 - 2 - 1 = -2$

17.6 Surface Integrals

- S19E2 #4



4. Find the area of the part of the plane $3x + 2y + z = 6$ that is in the first octant.

A. $3\sqrt{10}$
B. $3\sqrt{14}$
C. $3\sqrt{6}$
D. $3\sqrt{20}$
E. $3\sqrt{22}$

$u = x \qquad V = y \qquad z = 6 - 3u - 2v$

$$\int_0^2 \int_0^{3 - \frac{3}{2}u} \sqrt{14} \, dvdu = 3\sqrt{14}$$

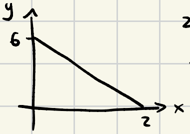
$r(t) = \langle x, v, 6 - 3u - 2v \rangle$

$r_u = \langle 1, 0, -3 \rangle$

$r_v = \langle 0, 1, -2 \rangle$

$r_u \times r_v = \begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = \langle 0 + 3, 2, 1 \rangle = \langle 3, 2, 1 \rangle$

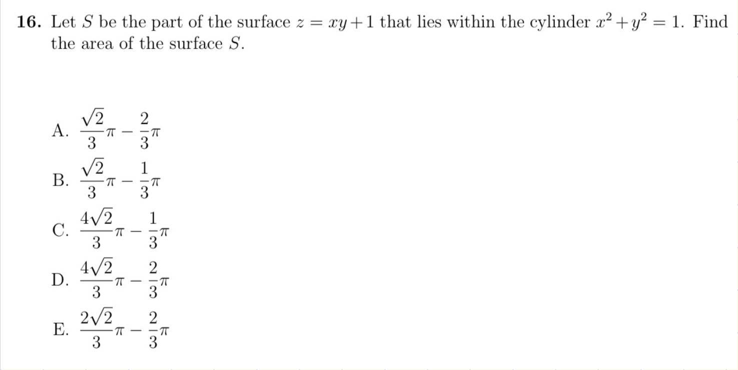
$|r_u \times r_v| = \sqrt{9 + 4} = \sqrt{14}$



$2y = 6 - 3x$

$y = 3 - \frac{3}{2}x$

· S19FE#16



16. Let S be the part of the surface $z = xy + 1$ that lies within the cylinder $x^2 + y^2 = 1$. Find the area of the surface S.

A. $\frac{\sqrt{2}}{3}\pi - \frac{2}{3}\pi$

B. $\frac{\sqrt{2}}{3}\pi - \frac{1}{3}\pi$

C. $\frac{4\sqrt{2}}{3}\pi - \frac{1}{3}\pi$

D. $\frac{4\sqrt{2}}{3}\pi - \frac{2}{3}\pi$

E. $\frac{2\sqrt{2}}{3}\pi - \frac{2}{3}\pi$

$$\iint \sqrt{z_x^2 + z_y^2 + 1}\, dA$$

$$\iint \sqrt{y^2 + x^2 + 1}\, dA$$

$$\int_0^{2\pi}\int_0^1 r\sqrt{1+r^2}\, dr d\theta$$

$u = 1+r^2$

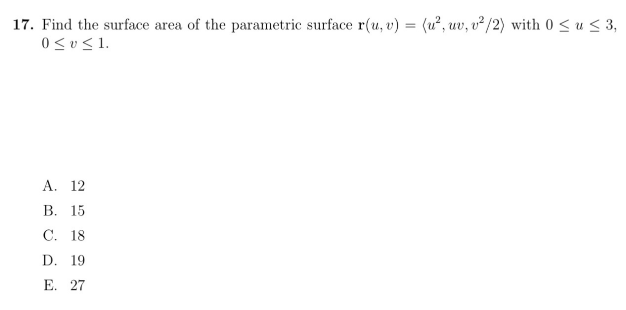
$du = 2r$

$$\frac{1}{2}\int_0^{2\pi}\int_1^2 u^{1/2} \qquad \frac{2}{3}u^{3/2}\Big|_1^2 = \frac{2}{3}(\sqrt{8} - 1)$$

$$\frac{1}{2}\int_0^{2\pi} \frac{2}{3}(\sqrt{8}-1) = \pi\,\frac{2}{3}(\sqrt{8}-1)$$

$$= \frac{2\pi}{3}(2\sqrt{2} - 1) = \frac{4\pi\sqrt{2}}{2} - \frac{2\pi}{3}$$

· S19FE#17



17. Find the surface area of the parametric surface $\mathbf{r}(u,v) = \langle u^2, uv, v^2/2\rangle$ with $0 \le u \le 3$, $0 \le v \le 1$.

A. 12

B. 15

C. 18

D. 19

E. 27

$$\iint |r_u \times r_v|\, du dv$$

$r_u = \langle 2u, v, 0\rangle$

$r_v = \langle 0, u, v\rangle$

$$r_u \times r_v = \begin{vmatrix} 2u & v & 0 \\ 0 & u & v \end{vmatrix} = \langle v^2, -2uv, 2u^2\rangle$$

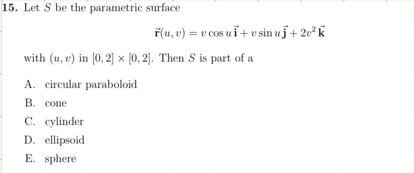
$$|r_u \times r_v| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(v^2+2u^2)^2} = v^2 + 2u^2$$

$$\int_0^3\int_0^1 v^2 + 2u^2\, dv du$$

$$\frac{1}{3}v^3 + 2u^2 v = \frac{1}{3} + 2u^2$$

$$\int_0^3 \frac{1}{3} + 2u^2\, du = \frac{1}{3}u + \frac{2}{3}u^3\Big|_0^3 = 1 + \frac{27 \times 2}{3} = 1 + 9(2) = 19$$

- S18FE#15



15. Let S be the parametric surface

$$\vec{\mathbf{r}}(u,v) = v\cos u\,\vec{\mathbf{i}} + v\sin u\,\vec{\mathbf{j}} + 2v^2\,\vec{\mathbf{k}}$$

with (u,v) in $[0,2]\times[0,2]$. Then S is part of a

A. circular paraboloid

B. cone

C. cylinder

D. ellipsoid

E. sphere

$V = r$

$U = \theta$

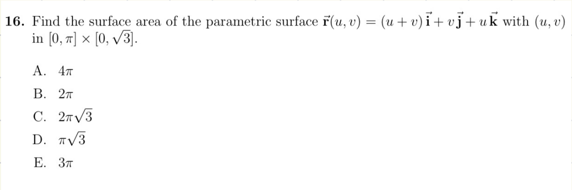
$x = r\cos\theta$

$V = r\sin\theta$

$z = 2r^2 = 2(x^2+y^2)$

Paraboloid : $z = x^2+y^2$

- S18FE#16



16. Find the surface area of the parametric surface $\vec{\mathbf{r}}(u,v) = (u+v)\,\vec{\mathbf{i}} + v\,\vec{\mathbf{j}} + u\,\vec{\mathbf{k}}$ with (u,v) in $[0,\pi]\times[0,\sqrt{3}]$.

A. 4π

B. 2π

C. $2\pi\sqrt{3}$

D. $\pi\sqrt{3}$

E. 3π

$r(u,v) = \langle u+v,\ v,\ u\rangle$

$r_u = \langle 1, 0, 1\rangle$

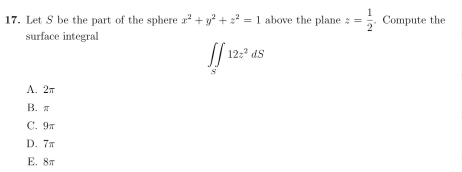
$r_v = \langle 1, 1, 0\rangle$

$r_u \times r_v = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, 1, 1\rangle$

$|r_u \times r_v| = \sqrt{3}$

$$\int_{0\,u}^{\pi}\int_{0\,v}^{\sqrt{3}} \sqrt{3}\,dv\,du = 3\pi$$

- S18FE#17



17. Let S be the part of the sphere $x^2+y^2+z^2=1$ above the plane $z=\frac{1}{2}$. Compute the surface integral

$$\iint_S 12z^2\,dS$$

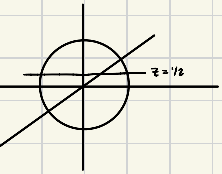
A. 2π

B. π

C. 9π

D. 7π

E. 8π



$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/3$

$\frac{1}{2} = \rho\cos\phi$

$\frac{1}{2} = \cos\phi$

$\phi = 60° = \pi/3$

$U = \phi \quad V = \theta$

$r(\phi,\theta) = \langle \rho\cos\theta\sin\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\phi\rangle$

$r_\phi = \langle \overset{a}{\rho\cos\theta\cos\phi},\ \overset{b}{\rho\sin\theta\cos\phi},\ \overset{c}{-\rho\sin\phi}\rangle$

$r_\theta = \langle \overset{d}{-\rho\sin\theta\sin\phi},\ \overset{e}{\rho\cos\theta\sin\phi},\ \overset{f}{0}\rangle$

$r_\phi \times r_\theta = \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = \langle bf - ce, -(af - cd), ae - bd\rangle$

$= \langle -ce,\ cd,\ ae - bd\rangle$

$= \langle \rho^2\cos\theta\sin^2\phi,\ \rho^2\sin\theta\sin^2\phi,\ \rho^2\cos^2\theta\sin\phi\cos\phi + \rho^2\sin^2\theta\sin\phi\cos\phi\rangle$

$$= \langle \rho^2\cos\theta\sin^2\phi,\ \rho^2\sin\theta\sin^2\phi,\ \rho^2\sin\phi\cos\phi \rangle$$

$$|r_\phi \times r_\theta| = \sqrt{\rho^4\cos^2\theta\sin^4\phi + \rho^4\sin^2\theta\sin^4\phi + \rho^4\sin^2\phi\cos^2\phi}$$

$$\rho^4\sin^4\phi + \rho^4\sin^2\phi\cos^2\phi$$

$$\rho^4\sin^2\phi\,(\sin^2\phi + \cos^2\phi)$$

$$= \sqrt{\rho^4\sin^2\phi} = \rho\sin\phi = \sin u$$

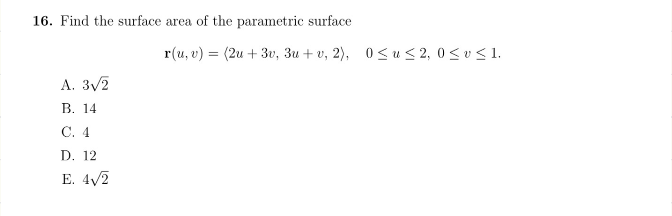
$$12\int_0^{2\pi}\int_0^{\pi/3} \rho^2\cos^2 u\ \sin u\ du\,dv = 12\int_0^{2\pi}\int_0^{\pi/3} \cos^2 u \sin u\ du\,dv$$

$$\underset{v}{} \quad \underset{u}{}$$

$a = \cos u$
$da = -\sin u$

$$-12\int_0^{2\pi}\int_1^{1/2} a^2\,da\,dv = \frac{-12}{3}\int_0^{2\pi} a^3\Big|_1^{1/2} \ (\tfrac{1}{8}-1) = \frac{12}{3}\times\frac{7}{8}\times 2\pi = \boxed{7\pi}$$

· F19FE # 16



16. Find the surface area of the parametric surface

$$\mathbf{r}(u,v) = \langle 2u+3v,\ 3u+v,\ 2\rangle, \quad 0 \le u \le 2,\ 0 \le v \le 1.$$

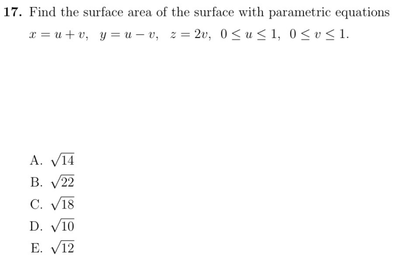
A. $3\sqrt{2}$
B. 14
C. 4
D. 12
E. $4\sqrt{2}$

$r_u = \langle 2, 3, 0\rangle$
$r_v = \langle 3, 1, 0\rangle$
$r_u \times r_v = \begin{vmatrix} 2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix} = \langle 0, 0, 2-9\rangle = \langle 0,0,-7\rangle$
$|r_u \times r_v| = \sqrt{49} = 7$

$$\int_0^2\int_0^1 7\ dv\,du = \boxed{14}$$

$u \quad v$

· F18FE # 17



17. Find the surface area of the surface with parametric equations

$x = u+v, \quad y = u-v, \quad z = 2v, \quad 0 \le u \le 1, \ 0 \le v \le 1.$

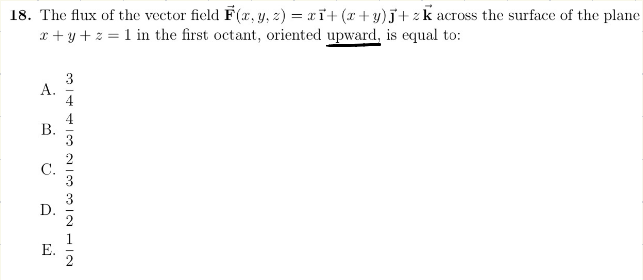
A. $\sqrt{14}$
B. $\sqrt{22}$
C. $\sqrt{18}$
D. $\sqrt{10}$
E. $\sqrt{12}$

$r(u,v) = \langle u+v,\ u-v,\ 2v\rangle$
$r_u = \langle 1, 1, 0\rangle$
$r_v = \langle 1, -1, 2\rangle$
$r_u \times r_v = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = \langle 2, -2, -1-1\rangle = \langle 2,-2,-2\rangle$
$|r_u \times r_v| = \sqrt{12}$

$$\int_0^1\int_0^1 \sqrt{12}\ du\,dv = \boxed{\sqrt{12}}$$

Vectors

· S18 FE #18



18. The flux of the vector field $\vec{\mathbf{F}}(x,y,z) = x\,\vec{\mathbf{\imath}} + (x+y)\,\vec{\mathbf{\jmath}} + z\,\vec{\mathbf{k}}$ across the surface of the plane $x+y+z=1$ in the first octant, oriented upward, is equal to:

A. $\frac{3}{4}$

B. $\frac{4}{3}$

C. $\frac{2}{3}$

D. $\frac{3}{2}$

E. $\frac{1}{2}$

For surface integral make $z=0$ and use floor trace

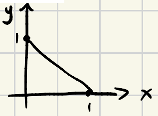
$\iint F\cdot\vec{n}\,dS$

$F = \langle x, x+y, z\rangle$

$n = \nabla G = \langle 1, 1, 1\rangle$

$G = x+y+z=1$

$z = 1-x-y$

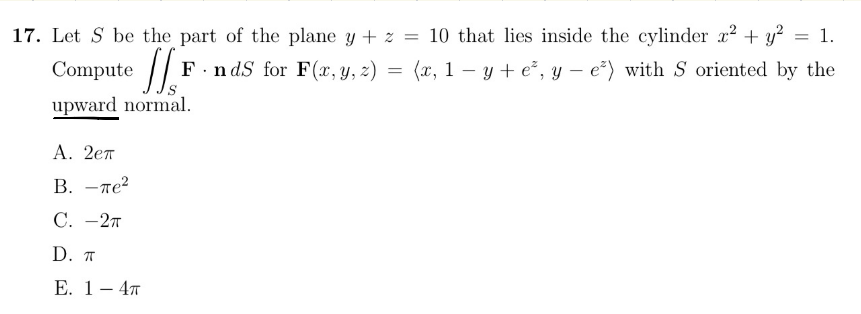


$$\int_0^1\int_0^{1-x} \langle x, x+y, 1-x-y\rangle\cdot\langle 1,1,1\rangle\,dy\,dx$$

$$\underset{x}{\int_0^1}\underset{y}{\int_0^{1-x}} x+1 = \int_0^1 (x+1)(1-x)\,dx$$

$$\int_0^1(-x^2+1)\,dx = -\tfrac{1}{3}x^3 + x\,\Big|_0^1 = \frac{2}{3}$$

· F19 FE #17



17. Let S be the part of the plane $y+z=10$ that lies inside the cylinder $x^2+y^2=1$. Compute $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ for $\mathbf{F}(x,y,z) = \langle x, 1-y+e^z, y-e^z\rangle$ with S oriented by the upward normal.

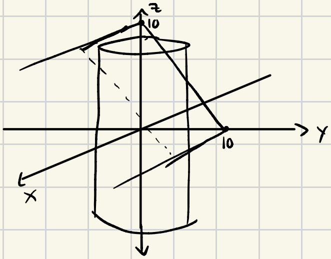
A. $2e\pi$

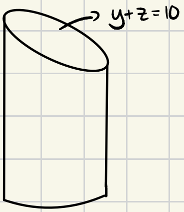
B. $-\pi e^2$

C. -2π

D. π

E. $1-4\pi$





$G = y+z-10$

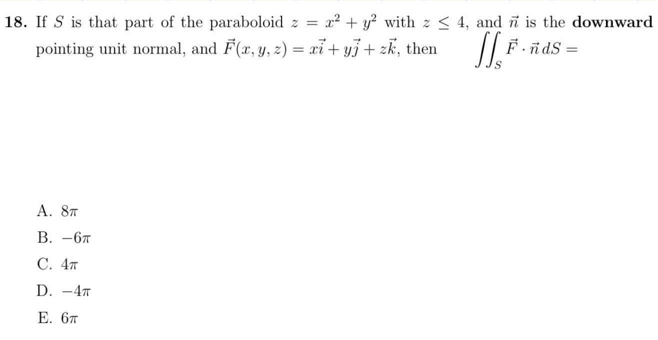
$n = \nabla G = \langle 0, 1, 1\rangle$

$\iint F\cdot n\,dS$

$$\underbrace{F\cdot n}_{\downarrow}\;\underbrace{\iint dS}_{\text{Circle area} = \pi r^2} = 1(\pi(1)^2) = \pi$$

$0+1-y+e^z+y-e^z=1$

· F18FE#18



18. If S is that part of the paraboloid $z = x^2 + y^2$ with $z \le 4$, and $\vec{n}$ is the **downward** pointing unit normal, and $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$, then $\iint_S \vec{F}\cdot\vec{n}\,dS =$

A. 8π

B. -6π

C. 4π

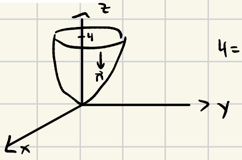
D. -4π

E. 6π

$F = \langle x, y, z \rangle$

$S = \langle x^2+y^2, -z \rangle$

$\nabla S = \langle 2x, 2y, -1 \rangle$



$4 = x^2+y^2 \quad r=2$

$$\iint \langle x,y,z\rangle \langle 2x, 2y, \overset{\text{Area}}{-1\rangle\, dS}$$

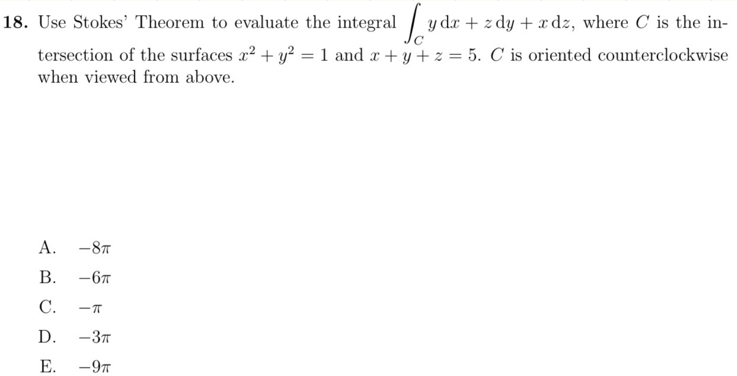
$$\iint 2x^2+2y^2-(x^2+y^2)\,dS$$

$$\iint x^2+y^2\,dS = \int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta$$

$$= \tfrac{1}{4}r^4\Big|_0^2 = 4\Big|_0^{2\pi} = 8\pi$$

17.7 Stoke's Theorem

·S19FE#18



18. Use Stokes' Theorem to evaluate the integral $\int_C y\,dx + z\,dy + x\,dz$, where C is the intersection of the surfaces $x^2+y^2=1$ and $x+y+z=5$. C is oriented counterclockwise when viewed from above.

A. -8π

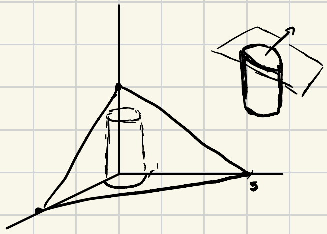
B. -6π

C. $-\pi$

D. -3π

E. -9π

$\iint_S \text{cur}\,F\cdot n\,dS = \oint_C F\cdot dr$

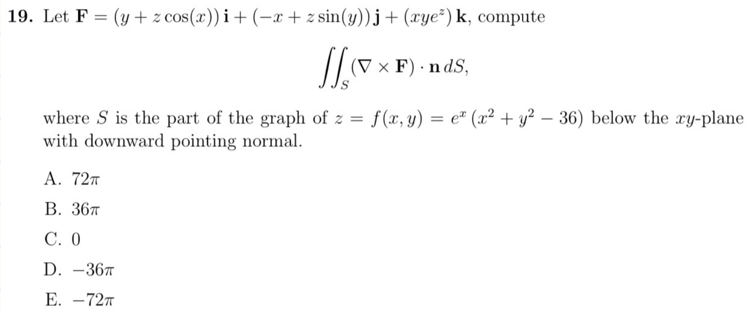


$F = \langle y, z, x \rangle \qquad n = \langle 1, 1, 1 \rangle$

$$\text{CurF} = \begin{vmatrix} \partial x & \partial y & \partial z \\ y & z & x \end{vmatrix} = \langle 0-1, -1, -1\rangle = \langle -1,-1,-1\rangle$$

$\iint curF\cdot\vec{n}ds = \iint -3\, ds$ $\iint ds = Area = \pi r^2 = \pi(1)^2 = \pi$ $\iint curF\cdot\vec{n}dS = -3\pi$

· F19FE#19



19. Let $\mathbf{F} = (y + z\cos(x))\,\mathbf{i} + (-x + z\sin(y))\,\mathbf{j} + (xye^z)\,\mathbf{k}$, compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where S is the part of the graph of $z = f(x,y) = e^x(x^2 + y^2 - 36)$ below the xy-plane with downward pointing normal.

A. 72π

B. 36π

C. 0

D. -36π

E. -72π

$\iint curF\cdot\vec{n}ds = \oint F\cdot dr$

$z=0= x^2+y^2-36$

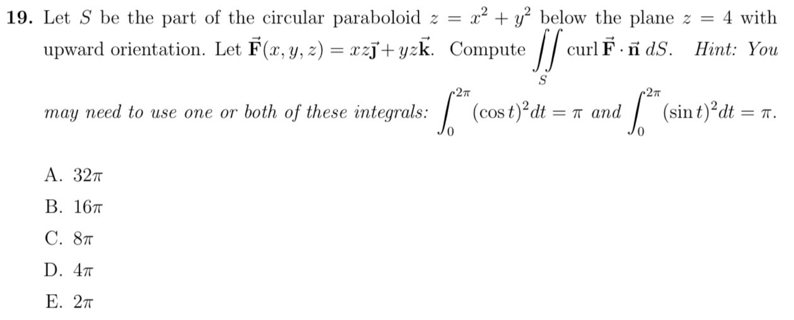
$r(t)=\langle 6\cos t, 6\sin t, 0\rangle$

$r(t)=\langle -6\cos t, -6\sin t, 0\rangle$

$r'(t)=\langle 6\sin t, -6\cos t, 0\rangle$

$\int_0^{2\pi} F\cdot dr = \int_0^{2\pi} 36\sin^2 t + 36\cos^2 t = 72\pi$

· S18FE#19



19. Let S be the part of the circular paraboloid $z = x^2 + y^2$ below the plane $z = 4$ with upward orientation. Let $\vec{\mathbf{F}}(x,y,z) = xz\vec{\mathbf{j}} + yz\vec{\mathbf{k}}$. Compute $\iint_S \operatorname{curl}\vec{\mathbf{F}}\cdot\vec{\mathbf{n}}\, dS$. *Hint: You may need to use one or both of these integrals:* $\int_0^{2\pi}(\cos t)^2 dt = \pi$ *and* $\int_0^{2\pi}(\sin t)^2 dt = \pi$.

A. 32π

B. 16π

C. 8π

D. 4π

E. 2π

$\oint F\cdot dr$:

$r(t)=\langle 2\cos t, 2\sin t, 4\rangle$

$r'(t)=\langle -2\sin t, 2\cos t, 0\rangle$

$F(t)=\langle 0, 8\cos t, 8\sin t\rangle$

$\oint F\cdot dr = \int_0^{2\pi} 16\cos^2 t = 16\pi$

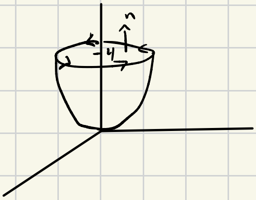
$\iint curF\cdot\vec{n}ds$:

$F=\langle 0, xz, yz\rangle$ $u=r$ $v=\theta$

$r(u,v)=\langle u\cos v, u\sin v, u^2\rangle$ $0\le r\le 2$

$r_u=\langle \overset{a}{\cos v}, \overset{b}{\sin v}, \overset{c}{2u}\rangle$ $0\le\theta\le 2\pi$

$r_v=\langle \overset{d}{-u\sin v}, \overset{e}{u\cos v}, \overset{f}{0}\rangle$



$r_u\times r_v = \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = \langle bf-ce, -(af-cd), ae-bd\rangle$

$= \langle 0-2u^2\cos v, -(0+2u^2\sin v), u\cos^2 v + u\sin^2 v\rangle$

$= \langle -2u^2\cos v, -2u^2\sin v, u\rangle$

$curF = \begin{vmatrix} \partial x & \partial y & \partial z \\ 0 & xz & yz \end{vmatrix} = \langle z-x, 0, z\rangle$

$= \langle u^2-u\cos v, 0, u^2\rangle$

$\langle u^2-u\cos v, 0, u^2\rangle \cdot \langle -2u^2\cos v, -2u^2\sin v, u\rangle$

$-2u^4\cos v + 2u^3\cos^2 v + 0 + u^3$

$= -2u^4\cos v + 2u^3\cos^2 v + u^3$

$\underset{v}{\int_0^{2\pi}} \underset{u}{\int_0^{2}} -2u^4\cos v + 2u^3\cos^2 v + u^3$

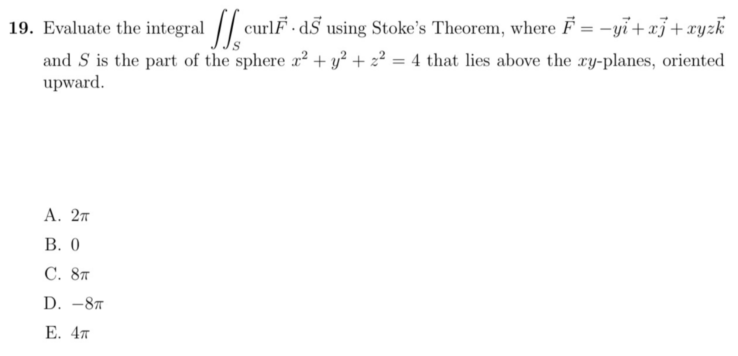
$-\frac{2}{5}u^5\cos v + \frac{2}{4}u^4\cos^2 v + \frac{1}{4}u^4 \Big|_0^2$

$-\frac{2}{5}(32)\cos v + \frac{2}{4}(16)\cos^2 v + \frac{16}{4}$

$\int_0^{2\pi} -\frac{2}{5}(32)\cos v + \frac{32}{4}\cos^2 v + 4$

$-\frac{2}{5}(32)\sin v \Big|_0^{2\pi} + 8\pi + 8\pi = 16\pi$

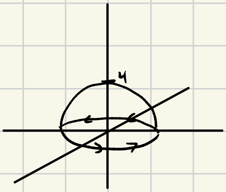
• F18FE # 19

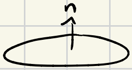


19. Evaluate the integral $\iint_S \text{curl}\vec{F} \cdot \text{d}\vec{S}$ using Stoke's Theorem, where $\vec{F} = -y\vec{i} + x\vec{j} + xyz\vec{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the xy-planes, oriented upward.

A. 2π
B. 0
C. 8π
D. -8π
E. 4π

$\oint \vec{F} \cdot d\vec{r}$





$r(t) = \langle 2\cos t, 2\sin t, 0 \rangle$

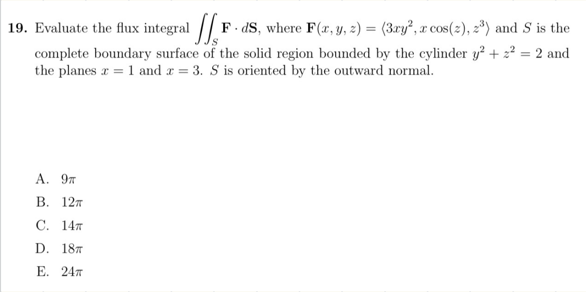
$r'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$

$F = \langle -y, x, xyz \rangle = \langle -2\sin t, 2\cos t, 0 \rangle$

$\int_0^{2\pi} 4\sin^2 t + 4\cos^2 t = 8\pi$

17.8 Divergence Theorem

• S19FE # 19

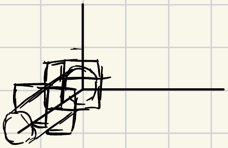


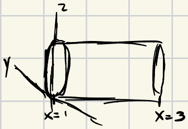
19. Evaluate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle 3xy^2, x\cos(z), z^3 \rangle$ and S is the complete boundary surface of the solid region bounded by the cylinder $y^2 + z^2 = 2$ and the planes $x = 1$ and $x = 3$. S is oriented by the outward normal.

A. 9π
B. 12π
C. 14π
D. 18π
E. 24π

$\iint F \cdot dS = \iiint \text{div} F \, dV$

$\text{div} F = 3y^2 + 0 + 3z^2$





$\underset{d\theta}{\int_0^{2\pi}} \underset{x}{\int_1^3} \underset{r}{\int_0^{\sqrt{2}}} 3r^3 \, dr\,dx\,d\theta = \frac{3}{4}r^4 \Big|_0^{\sqrt{2}} = 3$

$3(3-1) = 6$

$2\pi(6) = 12\pi$

· F19FE # 18

18. Consider $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$, where $\mathbf{r} = \langle x,\, y,\, z\rangle$ and $|\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2}$. Which one of the following is true

(i) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
(ii) $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 0$ for any closed surface S that encloses the origin.
(iii) $div(\mathbf{F}) = 0$.

A. None of the above.
B. Only (i) and (ii).
C. Only (i) and (iii)
D. Only (ii) and (iii).
E. All of the above.

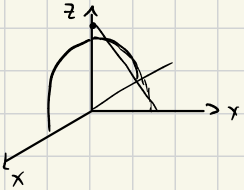
· F19FE # 20

20. Compute

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

the net outward flux of the vector field $\mathbf{F} = \langle x + y,\, y - z,\, xy + z\rangle$ across the surface S, which is the boundary of the solid bounded by $z = 0$, $y = 0$, $y + z = 2$, and $z = 1 - x^2$.

A. $-32/5$
B. $-32/15$
C. $-16/5$
D. $32/15$
E. $32/5$



$z = 2 - y \qquad z = 1 - x^2$

$2 - y = 1 - x^2$

$x^2 + 2 = y + 1$

$y = x^2 + 1$

$\mathrm{div}\, F = 3$

$$\underset{x}{\int_{-1}^{1}} \underset{z}{\int_{0}^{1-x^2}} \int_0^{2-z} 3 \; dy\, dz\, dx$$

$$\iint 6 - 3z \; dz\, dx$$

$$6z - \frac{3}{2}z^2 \Big|_0^{1-x^2}$$

$$6 - 6x^2 - \frac{3}{2}(1 - 2x^2 + x^4)$$

$$\int 6 - 6x^2 - \frac{3}{2} + 3x^2 - \frac{3}{2}x^4 \; dx$$

$$6x - 2x^3 - \frac{3}{2}x + x^3 - \frac{3}{10}x^5 \Big|_{-1}^{1}$$

$$\underbrace{(6 - 2 - \tfrac{3}{2} + 1 - \tfrac{3}{10})}_{\frac{16}{5}} - \underbrace{(-6 + 2 + \tfrac{3}{2} - 1 + \tfrac{3}{10})}_{\frac{16}{5}}$$

$$\frac{16}{5} + \frac{16}{5} = \frac{32}{5}$$

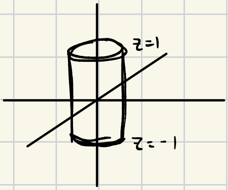
· S18FE # 20

20. Suppose $\vec{\mathbf{F}}(x, y, z) = 2xy^2\,\vec{\mathbf{\imath}} + 2yx^2\,\vec{\mathbf{\jmath}} - (x^2 + y^2)z\,\vec{\mathbf{k}}$ and S is the boundary surface of the solid enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$ and $z = 1$. S is a closed surface oriented by the outward normal. Calculate the flux integral $\displaystyle\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$.

A. 0
B. π
C. 2π
D. 3π
E. 4π

$$F = \langle 2xy^2,\ 2yx^2,\ -zx^2 - zy^2 \rangle$$

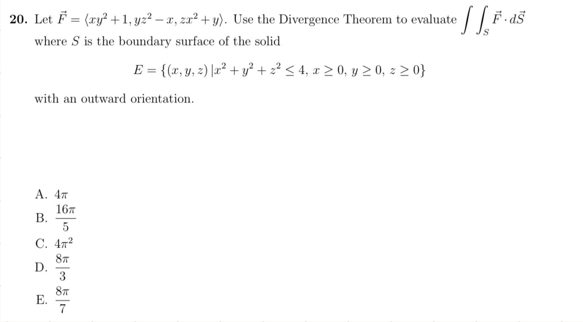
$$\text{div}F = 2y^2 + 2x^2 - x^2 - y^2 = y^2 + x^2$$



$$\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1} y^2+x^2 \, dr\, d\theta\, dz$$

$$\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1} r^3 \, dr\, d\theta\, dz = \iint \tfrac{1}{4}\, d\theta\, dz = \int \tfrac{1}{2}\pi\, dz = \tfrac{1}{2}\pi(1+1) = \pi$$

· F18FE # 20

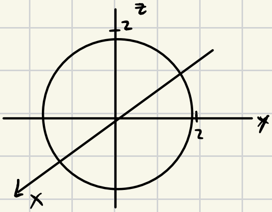


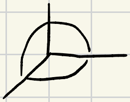
20. Let $\vec{F} = \langle xy^2+1, yz^2-x, zx^2+y\rangle$. Use the Divergence Theorem to evaluate $\int\int_S \vec{F}\cdot d\vec{S}$ where S is the boundary surface of the solid

$$E = \{(x,y,z)\,|\,x^2+y^2+z^2 \le 4,\ x \ge 0,\ y \ge 0,\ z \ge 0\}$$

with an outward orientation.

A. 4π
B. $\frac{16\pi}{5}$
C. $4\pi^2$
D. $\frac{8\pi}{3}$
E. $\frac{8\pi}{7}$





$$\text{div}F = y^2 + z^2 + x^2 = \rho^2$$

$$\int_{0}^{\pi/2}\int_{0}^{\pi/2}\int_{0}^{2} \rho^4 \sin\phi \, d\rho\, d\phi\, d\theta$$

θ φ ρ

$$\iint \tfrac{1}{5}\rho^5 \sin\phi$$

$$\int_{0}^{\pi/2} \tfrac{32}{5}\sin\phi = -\tfrac{32}{5}\cos\phi \Big|_{0}^{\pi/2}$$

$$-\tfrac{32}{5}(-1) = \tfrac{32}{5}$$

$$\int_{0}^{\pi/2} \tfrac{32}{5} = \frac{16\pi}{5}$$