Exam 1 (1-14)

1.1 DE and mathematical models

- Differential Equation: any equation that contains derivatives

- Satisfying DE:
  - If given a general solution manipulate it to address the derivatives
  - Solve for constant
  - Write solution in y= form

Ex) Find all solutions of $y''+y'-2y=0$ in form of $y=e^{rx}$ (r is some constant)

$y=e^{rx}$
$y'=re^{rx}$
$y''=r^2e^{rx}$

Plug in: $y''+y'-2y=0$
$r^2e^{rx}+re^{rx}-2(e^{rx})=0$
$e^{rx}(r^2+r-2)=0 \longrightarrow e^{rx}\neq 0$ for any x so $r^2+r-2=0 \longrightarrow$ $(r+2)(r-1)=0$, $r=-2$ and $r=1$

Solutions: $\boxed{y=e^{-2x} \text{ and } y=e^{x}}$

Note: Any constant multiple is also a solution

1.2 Integrals as General and particular Solutions

- General solution: contains "c" ∴ infinitely many solutions

  Ex) $y=x^2+c$

- Particular solution: Equation that doesn't have Variable representing a constant

  Ex) $y(0)=2$ $\quad y(x)=x^2+C$ $\quad$ so $\quad y=x^2+2$

Ex) $\frac{dy}{dx}=\frac{1}{x^2+4}$ $\quad y(0)=10$

Trig sub: $x=2\tan\theta \longrightarrow dx=2\sec^2\theta\, d\theta$
$\theta=\tan^{-1}(\frac{x}{2})$

$y=\int\frac{1}{4\tan^2\theta+4}dx \rightarrow y=\int\frac{1}{4(\tan^2+1)}\cdot 2\sec^2\theta\, d\theta \rightarrow y=\int\frac{2\sec^2\theta}{4\sec^2\theta} \longrightarrow y=\int\frac{1}{2}d\theta = \frac{1}{2}\theta+c$

$\boxed{y=\frac{1}{2}\tan^{-1}(\frac{x}{2})+c}$ General Solution

$y(0) = 10 \longrightarrow 10 = \frac{1}{2}\tan^{-1}(0) + C \longrightarrow C = 10 \longrightarrow \boxed{y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + 10}$ Particular solution

Trig sub:

$a^2 - x^2 \longrightarrow x = a\sin\theta$

$a^2 + x^2 \longrightarrow x = a\tan\theta$

$x^2 - a^2 \longrightarrow x = a\sec\theta$

1) Determine which sub is correct
2) Solve for x, dx, θ
3) Plug dx and x into equation and manipulate
4) Plug in for θ

Higher order:

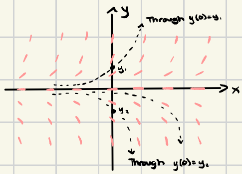
$a(t) = v'(t) = r''(t)$

$\int a(t) = v(t) = r'(t)$

$\iint a(t) = \int v(t) = r(t)$

1.3 Slope Field and Solution Curves

Ex) $\frac{dy}{dx} = y$



Slope is $\frac{dy}{dx} = y$

No dependency on x, moving left or right slopes stay the same

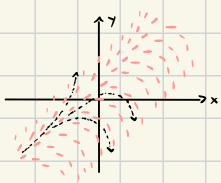
1) Where are slopes zero $(y' = y = 0)$
↳ along curve $y = 0$

2) above $y = 0 \rightarrow y > 0 \rightarrow y' = y > 0$ (positive slope: higher you go steeper the slope)

Guess solution: exponential, maybe $y = e^x$ or $y = -e^x$

Ex) $\frac{dy}{dx} = y - x$

Slopes depend on x and y

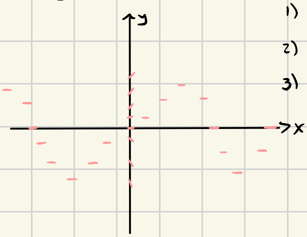


1) Start where $y' = 0$ $\qquad y' = y - x = 0 \rightarrow$ on curve $y = x$

2) Above it: $y > x \rightarrow y - x > 0$ so $y' > 0$ (positive slopes above

Notice: if $y(0) < 0$, as $x \rightarrow \infty$, $y \rightarrow -\infty$

Ex) $y' = y - \sin x$



1) $y' = 0$ on curve $y = \sin x$
2) Above $y = \sin x$ $\quad y' > 0$ (y inc $\rightarrow$ y' inc)
3) Below $y = \sin x$ $\quad y' < 0$ (y dec $\rightarrow$ y' dec)

If y fixed and move x-right

y fixed and move x-left

1.4 Separable Equations

Separable equations → we can always cleanly separate x and y
integrate both sides to solve

Ex) $\frac{dy}{dx} - xy = 0$

1) Rewrite: $\frac{dy}{dx} = xy$

2) Separate by dividing or multiplying: $\frac{1}{y}\,dy = x\,dx$

3) Integrate both sides: $\int \frac{1}{y}\,dy = \int x\,dx$

$\boxed{\ln|y| = \frac{1}{2}x^2 + C}$ Implicit form of solution

· Often can solve for y explicity

recall: $\ln a = b \leftrightarrow a = e^b$

$\ln|y| = \frac{1}{2}x^2 + C$ becomes : $|y| = e^{\frac{1}{2}x^2 + C}$

· Recall: $e^{a+b} = e^a \cdot e^b$

$|y| = e^{\frac{1}{2}x^2 + C} = e^{\frac{1}{2}x^2} \cdot e^C$ constant constant constant

e^C is constant call it C

e^C become C b/c $e \approx 2.7$ and C is some #.

some # × some # = some #
which is therefore a constant
and can be called C

$|y| = Ce^{\frac{1}{2}x^2}$

↳ often easy to find b/c we're given $y(x_0) = y_0$

· Drop | | b/c C can be of any sign

$\boxed{y = Ce^{\frac{1}{2}x^2}}$ Explicit form of solution

1.5 Linear First-Order Equations

$\frac{dy}{dx} + P(x)y = Q(x)$

↳ can't contain y

Integration factor: $I = e^{\int P(x)dx}$

Ex) $xy' + 3y = x$ $\quad y(1) = 2$

1) Rewrite: $y' + \frac{3y}{x} = 1$ (make coefficient of y', 1)

2) Identify $P(x)$ and $Q(x)$: $P(x) = \frac{3}{x}$ $\quad Q(x) = 1$

3) Integrating factor: $I = e^{\int P(x)dx} = e^{\int \frac{3}{x}dx} = e^{\ln(x)^3 + C} = x^3$ choose $C = 0$

4) Multiply equation by Integrating factor: $x^3\left(y' + \frac{3}{x}y\right) = 1(x^3)$

$\underbrace{x^3y' + 3x^2y} = x^3$

or multiply right side by I
left side $= yI\frac{d}{dx}$ then integrate

5) Determine the reverse of the product Rule $\left(\frac{d}{dx}Iy\right)$: $\frac{d}{dx}(x^3y) \rightarrow I = x^3 \rightarrow Iy = x^3y$

6) Integrate both sides: $\int x^3y\,\frac{d}{dx} = \int x^3 \rightarrow x^3y = \frac{1}{4}x^4 + C$ This C depends on initial condition

$y = \frac{1}{4}x + \frac{C}{x^3}$

7) Solve for C using inital condition: $y(1) = 2 = \frac{1}{4} + C \rightarrow C = \frac{7}{4}$

$\boxed{\text{So } y = \frac{1}{4}x + \frac{7}{4x^3}}$

Application: $\frac{dy}{dt}$ = (concentration in) (flow rate in) − (conc. out) (flow out)

$$= \left(\tfrac{1}{2}\,\text{lb/gal}\right)\left(4\,\text{gal/min}\right) - \left(\frac{y}{600}\,\text{lb/gal}\right)\left(4\,\text{gal/min}\right)$$

Salt in tank

Volume of water in tank

$$\frac{dy}{dt} = 2 - \left(\frac{y}{\text{Volume of water}}\right)(6)$$

→ $600 - 2t$

→ 4 in, 6 out, net loss of 2 (in − out)

1.6 Substitution Method and Exact Equations Part I

$\frac{dy}{dx} = f(ax + by + c)$ then $V = ax + by + C$ always works

Ex) $\frac{dy}{dx} = (3x - 2y + 5)^4$

homogeneous if $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

(make substitution $V = \frac{y}{x}$ then transform into separable and/or first-order linear)

Example: $y'(x+2y) = 6x - y$

1) Manipulate equation to get all variables in $\frac{y}{x}$ form

$$y' = \frac{(6x-y)\frac{1}{x}}{(x+2y)\frac{1}{x}} = \frac{6 - \frac{y}{x}}{1 + \frac{2y}{x}} \qquad V = \frac{y}{x}$$

2) Sub in v for $\frac{y}{x}$, find y', sub in for y'

$V = \frac{y}{x}$

$y = Vx$ $\qquad V + v'x = \frac{6 - V}{1 + 2V}$

$y' = V + v'x$

3) Isolate V to one side and write V' in $\frac{dv}{dx}$

$$v'x = \frac{6\ \ v}{1+2v} - v = \frac{6-v}{1+2v} - \frac{v(1+2v)}{(1+2v}$$

$v'x \qquad 6 \quad 2v \ -2v^2$

$$\frac{dv}{dx}x = \frac{6-2v-2v^2}{(1+2v)}$$

4) separate dx with x and dv with v then integrate

$$\frac{1+2v}{6-2v-2v^2}\,dv = \frac{1}{x}\,dx$$

$u = 6 - 2v - 2v^2$ $\qquad \ln|x| + c$

$du = -2 - 4v$

$= -2(1+2v)$

$$-\frac{1}{2}\int \frac{1}{u}\,du = \ln|x| + c$$

$$-\frac{1}{2}\ln|u| = \ln|x| + c$$

$$u = x^{-2}c$$

5) Unsub u and v

$$6 - 2v - 2v^2 = \frac{c}{x^2} \longrightarrow 6 - 2\frac{y}{x} - 2\left(\frac{y}{x}\right)^2 = \frac{c}{x^2}$$

$$6x^2 - 2yx - 2y^2 = C$$

Bernoulli Differential Equation

Always want coefficient of y' to = 1

$$y' + P(x)y = Q(x)y^n \qquad \left.\begin{matrix} n \neq 0 \\ n \neq 1 \end{matrix}\right\}$$ (These make the DE separable or linear)

1) Substitution: $V = y^{1-n}$
2) Solve for y then y' (use product rule)
3) Solve for V
4) Undo substitution

Example: $2x^2y' + 6xy = 14y^3$

1) convert to Bernoulli format $\quad y' + \frac{3y}{x} = \frac{7y^3}{x^2}$

2) Find y':

$V = y^{1-n} = y^{-2}$

$V' = -2y^{-3} \cdot y'$

$y' = \frac{-V'y^3}{2}$

3) sub in for y' only $\quad \frac{-V'y^3}{2} + \frac{3y}{x} = \frac{7y^3}{x^2}$

4) Isolate V': $\quad V' + \frac{-6}{\underbrace{y^2}_{v} x} = \frac{-14}{x^2}$

5) Use 1st order $\quad V' + \frac{-6}{Vx} = \frac{-14}{x^2}$

$I = e^{\int -\frac{6}{x}} = e^{-6\ln|x|} = x^{-6}$

$x^{-6}\left(V' + \frac{-6}{y^2 x}\right) = \frac{-14x^{-6}}{x^2}$

$x^{-6}V = \int \frac{-14}{x^8}\,dx$

$x^{-6}V = \frac{2}{x^7} + C$

6) Undo V-sub $\quad V = \frac{2}{x} + Cx^6$

$y^{-2} = \frac{2}{x} + Cx^6$

$y^{-2} - \frac{2}{x} = Cx^6$

$\frac{1}{y^2x^6} - \frac{2}{x^7} = C$

Substitution

$M\,dx + N\,dy = 0$ and $M_y = N_x$ } Exact Differential Equation

Example: $\underbrace{(3x^2 + 2y^2)}_{M}dx + \underbrace{(4xy + 6y^2)}_{N}dy = 0$ $\qquad$ Process for Exact

1) Test if exact ($M_y = N_x$)

$4y = 4y$ ✓

so ∴ there's a function $f(x,y)$ such that $\quad f_x = M = 3x^2 + 2y^2 \quad f_y = N = 4xy + 6y^2$

2) Recover f by picking M or N and integrating

$\frac{\partial f}{\partial x} = \underbrace{3x^2 + 3y^2}_{M} \longrightarrow f = \int (3x^2 + 2y^2)\,dx$ → ∴ y is constant

$= x^3 + 2xy^2 + g(y)$

3) Find $g(y)$ by taking partial with respect to y and compare to N (must be equal to N)

From f we got from integration

$f_y = \underbrace{4xy + g'(y)}_{\text{From } f \text{ we got from integration}} = \overbrace{4xy + 6y^2}^{N}$

$g'(y) = 6y^2 \quad$ so $\quad g(y) = 2y^3 + C_1$

4) Plug $g(y)$ back into $f(x,y)$

$f(x,y) = x^3 + 2y^3x + 2y^3 + C_1$

Solution of this exact DE is $f(x,y) = C$

$f(x,y) = x^3 + 2y^2x + 2y^3 + C_1 = C$

$$\boxed{x^3 + 2y^2x + 2y^3 = C}$$

Characteristic Polynomials (For second order DE)

→ Note: Cov'd in exam 2 but easier way to do second order

General rule: $y'' + ay' + by = 0$

↳ Solution: $y = e^{rx}$

$y'' + 5y' + 4y = 0$

↓ ↓ ↓ Carry down constant derivative is power

$r^2 + 5r + 4 = 0$

$(r+4)(r+1) = 0$ Solve for r

$r = -4 \quad r = -1$

$$\boxed{y = C_1e^{-4x} + C_2e^{-x}}$$

If you get $r^2 + C = 0 \longrightarrow r = \pm i \longrightarrow$ $e^{ix} = \cos x + i\sin x$
$e^{-ix} = \cos x - i\sin x$

So: $y_1 = \cos x + i\sin x$

$y_2 = \cos x - i\sin x$

$y = C_1(\cos x + i\sin x) + C_2(\cos x - i\sin x)$

$= (C_1 + C_2)\cos x + i(C_1 - C_2)\sin x$

$= A\cos x + B\sin x$

If: $r = \pm 8i$

Then: $A\cos(8x) + B\sin(8x)$

- Plug in initial condition y and y' solve for A and B then re-write y-equation

If y is not present

Ex) $xy + y' = x$

Sub: $P = \frac{dy}{dx} = y'$

$\frac{dP}{dx} = y''$

Try it: $x\frac{dP}{dx} + P = x$ 1st-order in P and x

$\frac{dP}{dx} + \frac{1}{x}P = 1$ Linear w/ P coefficients on x and/or numbers

$I = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$x\left(\frac{dP}{dx} + \frac{1}{x}P\right) = x$

$x\frac{dP}{dx} + P = x$

$\frac{d}{dx}(xP) = x \longrightarrow xP = \frac{1}{2}x^2 + C$

$P = \frac{1}{2}x + \frac{C}{x}$

Recover y:

$P = \frac{dy}{dx} = \frac{1}{2}x + \frac{C}{x}$

$y = \int\left(\frac{1}{2}x + \frac{C}{x}\right)dx$

constant

$\boxed{y = \frac{1}{4}x^2 + C\ln(x) + D}$