MA 266 Lecture 13

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Sec 3.1 Second Order Linear Equations

- Recall that a second-order differential equation in the (unknown) function $y(x)$ is:

- This differential equation is said to be *linear* provided that G is linear in the dependent variable y and its derivatives y' and y''.
- Thus a linear second-order equation takes the form:

- We assume that the (known) coefficient functions $A(x)$, $B(x)$, $C(x)$, and $F(x)$ are continuous on some open interval I.

Defintion 1. *A __________________ linear equation takes the form:*

that is __________________.

- If __________________, the linear equation is _________________________.

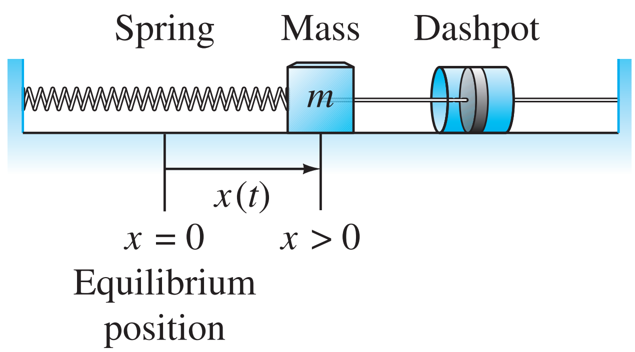
Example 1. *Homogeneous vs. Nonhomogeneous*

a) $x^2y'' + 2xy' + 3y - \cos x = 0$

b) $x^2y'' + 2xy' + 3y = 0$

- In case the differential equation models a physical system, the nonhomogeneous term $F(x)$ frequently corresponds to some *external* influence on the system.

Example 2. *Model the following mass-spring-dashpot system using linear equations.*



- ___________________ mass attached to a spring and a dashpot (shock absorber).

- ___________________ force of the spring on the mass

- ___________________ force of the dashpot on the mass

- Assume the restoring force F_S of the spring is proportional to the *displacement* x of the mass from its equilibrium position:

- Assume the dashpot force F_R is proportional to the *velocity* $v = dx/dt$ of the mass and acts opposite to the direction of motion:

- Newton's law $F = ma$ gives

- The *homogeneous* linear equation is then:

- If, in addition to F_S and F_R, the mass m is acted on by an external force $F(t)$, the resulting equation is

Homogeneous Second-Order Linear Equations

- Consider the general second-order linear equation

$$A(x)y'' + B(x)y' + C(x)y = F(x),$$

- If ________________________, we can write the above equation in the form:

- The corresponding *homogeneous* equation:

Theorem 1 Principle of Superposition for Homogeneous Equations

- Let y_1 and y_2 be two solutions of the homogeneous linear equation ____________________
- If c_1 and c_2 are constants, then the linear combination

  is also a solution of this equation on I.

Why the Theorem 1 is true?

- Note that the linearity of differentiation gives

- Then because y_1 and y_2 are solutions,

- Thus

Theorem 2 Existence and Uniqueness for Linear Equations

- Suppose that the functions p, q, and f are continuous on the open interval I containing the point a.
- Then, given any two numbers b_0 and b_1, the equation

  $$y'' + p(x)y' + q(x)y = f(x)$$

  has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

Remark

- The differential equation and the initial conditions in the theorem constitute a second-order linear *initial value problem*.
- Theorem 2 tells us that any such initial value problem has a unique solution on the *whole* interval I where the coefficient functions in the equation are continuous.

Example 3. *Consider the following homogeneous second-order linear equation:*

$$x^2y'' - 2xy' + 2y = 0.$$

Let $y_1 = x$ *and* $y_2 = x^2$. *Find a solution of the form* $y = c_1y_1 + c_2y_2$ *that satisfies the following initial conditions:*

$$y(1) = 3 \text{ and } y'(1) = 1.$$

- Where the *unique* solution exists?
- Using the given initial conditions:

Ensuring That the Equations Have a Solution

- In order for the procedure of the previous example to succeed, the two solutions y_1 and y_2 must have the property that the equations

  can always be solved for c_1 and c_2, no matter what the initial conditions b_0 and b_1 might be.

Defintion 2. *Two functions defined on an open interval I are said to be* __________________ *on I provided that neither is a constant multiple of the other.*

Linear Dependence

- Two functions are said to be __________________ on an open interval provided one of them is a constant multiple of the other.

- We can determine whether two given functions f and g are linearly dependent on an interval I by noting whether either of the quotients f/g or g/f is a constant-valued function on I.

Example 4. *Determine if the following pair of functions are independent.*

$$\begin{aligned} \sin x \quad &\textit{and} \quad \cos x; \\ e^x \quad &\textit{and} \quad e^{-2x}; \\ \sin 2x \quad &\textit{and} \quad \sin x \cos x. \end{aligned}$$

General Solution

- We want to show, finally, that given *any* two linearly independent solutions y_1 and y_2 of the homogeneous equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

every solution y of the equation

$$y'' + py' + qy = 0$$

can be expressed as a linear combination

$$y = c_1y_1 + c_2y_2$$

of y_1 and y_2.

The Wronksian

- As suggested by the equations

$$c_1y_1(a) + c_2y_2(a) = b_0,$$
$$c_1y_1'(a) + c_2y_2'(a) = b_1,$$

the determination of the constants c_1 and c_2 in depends on a certain 2×2 determinant of values of y_1, y_2, and their derivatives.

- Given two functions f and g, the __________________________ of f and g is:

Example 5. *Compute the Wroskian of* $f(x) = \cos x$ *and* $g(x) = \sin x$.

- The Wroskian of two linearly *dependent* functions is zero:

Theorem 3 General Solutions of Homogeneous Equations

- Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

  with p and q continuous on the open interval I.

- If Y is any solution of this equation on I, then there exist numbers c_1 and c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

  for all x in I.

Linear Second-Order Equations with Constant Coefficients

- Consider the homogeneous second-order linear differential equation

  with *constant* coefficients __________________.

- Consider the *ansatz*: __________________.
- By noting:

- We conclude that __________________ satisfy

- when ____________ is a *root* of the *algebraic* equation:

- This quadratic equation is called the ________________________ of the homogeneous linear differential equation.

Characteristic Roots

- If the algebraic equation

$$ar^2 + br + c = 0$$

has two ________________________ roots r_1 and r_2, then the corresponding solutions:

are *linearly independent.*

- Theorem 3 then implies that

- is a *general* solution of __________________.
- This leads to the following theorem.

Theorem 4 Distinct Real Roots

- If the roots r_1 and r_2 of the characteristic equation

$$ar^2 + br + c = 0$$

are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a general solution of

$$ay'' + by' + cy = 0.$$

Equal Roots

- If the characteristic equation

$$ar^2 + br + c = 0$$

  has equal roots $r_1 = r_2$, we get (at first) only the single solution __________________ of the differential equation

$$ay'' + by' + cy = 0.$$

- The problem in this case is to produce the “missing” second solution of the differential equation.

- A double root $r = r_1$ will occur precisely when the characteristic equation is a constant multiple of the equation

- Any differential equation with this characteristic equation is equivalent to

- But it is easy to verify that __________________ is a second solution of ____________.

- Moreover, it is easy to check that

$$y_1(x) = e^{r_1 x} \quad \text{and} \quad y_2(x) = xe^{r_1 x}$$

  are linearly independent functions, so by Theorem 3, the general solution of the differential equation

$$y'' - 2r_1 y' + r_1^2 y = 0$$

  is

Theorem 5 Repeated roots

- If the characteristic equation

$$ar^2 + br + c = 0$$

has equal (necessarily real) roots $r_1 = r_2$, then

$$y(x) = (c_1 + c_2 x)e^{r_1 x}$$

is a general solution of the differential equation

$$ay'' + by' + cy = 0.$$

Example 6. *Find the general solution of the differential equation:*

$$y'' + 2y' - 15y = 0$$

Example 7. *Find the general solution of the differential equation:*

$$9y'' - 12y' + 4y = 0$$

Example 8. *Let* $y(x) = c_1 + c_2 e^{-10x}$ *be a general solution of a homogeneous second-order differential equation of the form*

$$ay'' + by' + c = 0,$$

with constant coefficients. Find such coefficients.