RESEARCH STATEMENT

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1 Introduction

The *partitions* of $n \in \mathbb{N}$ are tuples of positive integers $(a_1, a_2, \ldots, a_k)$ such that $a_1 \geq a_2 \geq \ldots \geq a_k$ and $a_1 + a_2 + \cdots + a_k = n$. Using $\Pi[n]$ to denote the set of such partitions, the quantities $\mathfrak{p}(n) := \mathrm{card}(\Pi[n])$ are the (ordinary) *partition numbers*. In 1918 Hardy and Ramanujan used the functional equation of the Dedekind η-function to give an asymptotic formula for $\mathfrak{p}(n)$ as $n \to \infty$; namely, in [13] they establish (in addition to stronger results) that

$$\log \mathfrak{p}(n) \sim \kappa\sqrt{n}, \qquad \text{where } \kappa := \pi\sqrt{2/3}, \tag{1.1}$$

and the relation $a(n) \sim b(n)$ indicates that $\lim_{n\to\infty} a(n)/b(n) = 1$. At nearly the same time Ramanujan announced and proved his eponymous "congruences", which are the relations that

$$\mathfrak{p}(5n+4) \equiv 0 \pmod 5, \quad \mathfrak{p}(7n+5) \equiv 0 \pmod 7, \quad \text{and} \quad \mathfrak{p}(11n+6) \equiv 0 \pmod{11}.$$

Recent years have seen a surge of interest in partition theory led by researchers including B. Berndt, A. Malik, R. C. Vaughan, and A. Zaharescu (see, e.g., [3, 10, 19, 20]). Here we discuss our work on both arithmetic and analytic aspects of the novel class of "signed" partition enumerations involving multiplicative $f : \mathbb{N} \to \{0, \pm 1\}$. Following this, we discuss a few problems toward developing more general results on some families of signed partition numbers.

2 Past Work

Let $f : \mathbb{N} \to \{0, \pm 1\}$, and for $n \in \mathbb{N}$ and any partition $\pi = (a_1, a_2, \ldots, a_k) \in \Pi[n]$ let

$$f(\pi) := f(a_1)f(a_2)\cdots f(a_k). \tag{2.1}$$

With this we define the *signed partition numbers*

$$\mathfrak{p}(n, f) = \sum_{\pi \in \Pi[n]} f(\pi). \tag{2.2}$$

Definition (2.2) generalizes several classical partition-related quantities, e.g., with the constant function 1 one has $\mathfrak{p}(n) = \mathfrak{p}(n, 1)$, and with the indicator function $\mathbf{1}_A$ for $A \subset \mathbb{N}$, the quantities $\mathfrak{p}(n, \mathbf{1}_A)$ are the *A-restricted partition numbers*. Many examples and families of restricted partitions numbers are well studied (see, e.g., [1, 12]).

2.1 **Arithmetic work; q-series and periodic vanishings.** We say a sequence $(a_n)_{\mathbb{N}}$ *vanishes on an arithmetic progression* (or has a *periodic vanishing*) if one has $a_{jm+r} = 0$ for some $0 \le r < m$ and all $j \ge 0$. In such a case we may say that a_n "vanishes on all $n \equiv r \pmod{m}$". The past two decades have seen a boom of work on periodic vanishings in the coefficients of various q-series; we note [4, 5, 14, 16, 18] as only a small portion of the extant literature.

Two 10-periodic vanishings in sequences $(\mathfrak{p}(n, f))_{\mathbb{N}}$ related to the Legendre symbol $\chi_5(n) := (\frac{n}{5})$ were recently discovered by the author. In addition to $\chi_5(\pi)$, defined via (2.1), for any partition $\pi = (a_1, a_2, \ldots, a_k)$ of any positive n we set

$$\chi_5^\dagger(\pi) := (-1)^k \chi_5(\pi) = (-1)^k \chi_5(a_1)\chi_5(a_2)\cdots\chi_5(a_k).$$

One of the primary results of [8] is given in the following theorem.

Theorem 2.1. *One has that*

$$\begin{aligned} \mathfrak{p}(n, \chi_5) &= 0 \qquad \text{for } n \equiv 2 \pmod{10}, \\ \mathfrak{p}(n, \chi_5^\dagger) &= 0 \qquad \text{for } n \equiv 6 \pmod{10}. \end{aligned}$$

In [8], Theorem 2.1 is proved using the theory of q-*series* identities and extensive symbolic manipulations. A "soft" explanation of the periodic vanishing demonstrated by $\mathfrak{p}(n, \chi_5)$ is provided by the following asymptotic formula.

Theorem 2.2 ([9, Thm. 1.7])**.** *As $n \to \infty$ one has*

$$\mathfrak{p}(n, \chi_5) = \mathfrak{a}_5 n^{-3/4} \exp\Big(\tfrac{1}{2}\kappa\sqrt{\tfrac{4}{5}n}\Big)\big[1 + (-1)^n \mathfrak{b}_5 + \mathfrak{d}_5 \cos\big(\tfrac{2\pi}{5}n - \tfrac{\pi}{10}\big) + O(n^{-1/5})\big], \tag{2.3}$$

where

$$\kappa = \pi\sqrt{\tfrac{2}{3}}, \quad \mathfrak{a}_5 = \left(\frac{3+\sqrt{5}}{960}\right)^{1/4}, \quad \mathfrak{b}_5 = \frac{3-\sqrt{5}}{2}, \quad \textit{and} \quad \mathfrak{d}_5 = \sqrt{2(5-\sqrt{5})}.$$

Ignoring the error term $O(n^{-1/5})$ in (2.3) and considering the 10-periodic term

$$\mathfrak{S}(n) := 1 + (-1)^n\left(\frac{3-\sqrt{5}}{2}\right) + \sqrt{2(5-\sqrt{5})}\,\cos\left(\frac{2\pi n}{5} - \frac{\pi}{10}\right),$$

it is surprising to find that

$$\mathfrak{S}(2) = 0 \qquad \text{and} \qquad \mathfrak{S}(n) \neq 0 \quad \text{for } 1 \le n \le 10 \text{ with } n \neq 2.$$

This provides a soft explanation for the periodic vanishing of $\mathfrak{p}(n, \chi_5)$. The surprising nature of this periodic vanishing is amplified by the following further result.

Theorem 2.3 ([9, Thm. 1.10])**.** *For odd primes p, let $\chi_p = \chi_p(n)$ denote the Legendre symbol $(\frac{n}{p})$. If $p \neq 5$ and $p \not\equiv 1 \pmod 8$, then the sequence $(\mathfrak{p}(n, \chi_p))_{\mathbb{N}}$ does not vanish on any arithmetic progression.*

2.2 **Asymptotic results on some $\mathfrak{p}(n,f)$.** When f assumes both positive and negative values one expects these signs to cause cancellations in the sums $\mathfrak{p}(n,f)$. We recall the Möbius μ and Liouville λ functions from prime number theory: If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ with distinct primes p_i and all $a_i \geq 1$, then

$$\lambda(n) := (-1)^{a_1+\cdots+a_r} \qquad \text{and} \qquad \mu(n) := \begin{cases} (-1)^r & \text{if all } a_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The following result is an immediate corollary of the main results of [7].

Theorem 2.4. *For all $\varepsilon > 0$, as $n \to \infty$ one has*

$$\mathfrak{p}(n,\mu) = O\big(e^{(1+\varepsilon)\sqrt{n}}\big) \qquad \textit{and} \qquad \mathfrak{p}(n,\lambda) = O\big(e^{(\frac{1}{2}+\varepsilon)\kappa\sqrt{n}}\big), \tag{2.4}$$

where $\kappa = \pi\sqrt{2/3}$. In addition, for positive integer k, as $k \to \infty$ one has

$$\log \mathfrak{p}(2k,\mu) \sim \sqrt{2k} \qquad \textit{and} \qquad \log \mathfrak{p}(2k,\lambda) \sim \tfrac{1}{2}\kappa\sqrt{2k}. \tag{2.5}$$

Given the relations of (2.5), it is natural to consider to what extent those relations "extend" to odd n. In [6] this question is answered under mild assumptions on the zeros of the Riemann zeta function $\zeta(s)$. Let $\Theta := \sup\{\mathrm{Re}(\rho) : \zeta(\rho) = 0\}$. It is well known that $\frac{1}{2} \leq \Theta \leq 1$; the assertion that $\Theta = \frac{1}{2}$ is the Riemann Hypothesis (RH).

Again for odd primes p let $\chi_p(n)$ denote the Legendre symbol $(\frac{n}{p})$. In [9], asymptotic formulae for different families of *Legendre-signed partition numbers* $\mathfrak{p}(n,\chi_p)$ are established, where primes are separated by their residue modulo 8.

Theorem 2.5 ([9, Thm. 1.3])**.** *Let p be an odd prime such that $p \neq 5$ and $p \equiv 1 \pmod 4$, and let $L(s,\chi_p)$ be the Dirichlet L-function for χ_p. As $n \to \infty$ one has*

$$\mathfrak{p}(n,\chi_p) = \mathfrak{a}_p n^{-3/4} \exp\Big(\tfrac{1}{2}\kappa\sqrt{(1-\tfrac{1}{p})n}\Big)\big[1 + (-1)^n \mathfrak{b}_p + O(n^{-1/5})\big],$$

where

$$\kappa = \pi\sqrt{2/3}, \qquad \mathfrak{a}_p = \big(\tfrac{p-1}{384p^2}\big)^{\frac{1}{4}} \exp(\tfrac{1}{4}\sqrt{p}L(1,\chi_p))$$

and

$$\mathfrak{b}_p = \begin{cases} 1 & p \equiv 1 \pmod 8, \\ \exp(-\sqrt{p}L(1,\chi_p)) & p \equiv 5 \pmod 8 \textit{ and } p \neq 5. \end{cases}$$

The corresponding asymptotic formulae for $p \equiv 3 \pmod 4$ are similar to those of Theorem 2.5 but involve far more complicated constants. As such, we present the following simpler asymptotic result, which is Corollary 1.6 in [9].

Theorem 2.6. *If $p \equiv 7 \pmod 8$, then as $n \to \infty$ one has*

$$\mathfrak{p}(n,\chi_p) \asymp n^{\sqrt{p}L(1,\chi)/4\pi - 3/4} \exp\Big(\tfrac{1}{2}\kappa\sqrt{(1-\tfrac{1}{p})n}\Big).$$

If $p \equiv 3 \pmod 8$, then as $n \to \infty$ one has the stronger relation

$$\mathfrak{p}(n,\chi_p) \sim \mathfrak{a}_p n^{\sqrt{p}L(1,\chi)/4\pi - 3/4} \exp\Big(\tfrac{1}{2}\kappa\sqrt{(1-\tfrac{1}{p})n}\Big),$$

where

$$\mathfrak{a}_p = \left(\frac{p-1}{384\,p^2}\right)^{\frac{1}{4}} \exp\left(\frac{\sqrt{p}L(1,\chi)}{2\pi}\left(\gamma + \frac{1}{2}\log\left(\frac{384}{p(p-1)}\right) - \frac{L'(1,\chi)}{L(1,\chi)}\right)\right),$$

where γ is the Euler-Mascheroni constant.

3 Proposed research

The novelty of the results of Theorems 2.2 and 2.3 indicate that sequences $\mathfrak{p}(n,\chi_p)_{\mathbb{N}}$ involving primes $p \equiv 1 \pmod 8$ should be further explored. Basic empirical computations have been done, but these computations further indicate that the periodic vanishing of $\mathfrak{p}(n,\chi_5)$ is indeed quite rare.

Problem 3.1. Establish the presence of periodic vanishings, or lack thereof, in sequences $(\mathfrak{p}(n,\chi_p))_{\mathbb{N}}$ where $p \equiv 1 \pmod 8$.

When $p \equiv 1 \pmod 4$, the generating function

$$\prod_{r=1}^{p-1}(\chi_p(r)q;q^p)_\infty^{-1} = 1 + \sum_{n=1}^{\infty}\mathfrak{p}(n,\chi_p)q^n$$

can be expressed as a quotient of $\eta(q)$ and of Jacobi θ-functions, and thus enjoys a modular-like functional equation. This functional transformation relation allows one to give a convergent series representation for the coefficients $\mathfrak{p}(n,\chi_p)$, à-la Rademacher's convergent series for the ordinary partition numbers $\mathfrak{p}(n,1)$.

Specifically, using the abbreviations $\kappa = \pi\sqrt{2/3}$ and $\lambda_n = \sqrt{n-1/24}$, Rademacher establishes [17] that

$$\mathfrak{p}(n) = \kappa(384)^{-\frac{1}{4}}\lambda_n^{\frac{3}{4}}\sum_{k=1}^{\infty}A_k(n)k^{-1}I_{\frac{3}{2}}(\kappa\lambda_n/k), \tag{3.1}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind, and $A_k(n)$ is related to the classical Kloosterman sums.

For fixed $p \equiv 1 \pmod 4$ let

$$\mathfrak{L}_k(n) := \sum_{0<h\le k}{}' \exp\{\pi i\Lambda(h,k) - 2\pi i hn/k\}, \tag{3.2}$$

where $\Lambda(h,k)$ is a certain "character-twisted" Dedekind sum. Specifically,

$$\Lambda(h,k) = \tfrac{1}{2}\{s_\chi(h,k) - s_\chi(2h,k)\} + \tfrac{1}{2}\{s(2h,k) - s(2hp,k)\}, \tag{3.3}$$

$$s_\chi(h,k) := \sum_{\mu \bmod [k,p]} \chi(\mu)(\!(h\mu/k)\!)(\!(\mu/[k,p])\!), \tag{3.4}$$

where $[k,p] = \mathrm{lcm}(k,p)$, and $(\!(x)\!) = 0$ for $x \in \mathbb{Z}$ and $(\!(x)\!) = x - [x] - \frac{1}{2}$ otherwise. We note that when $(k,p) = 1$, our $s_\chi(h,k)$ agrees with the $s_\chi(h,k)$ in Berndt's notation [2].

Considering Rademacher's formula (3.1), it is clear that, hypothetically, $\mathfrak{p}(n)$ would vanish if all $A_k(n) = 0$; this is, of course, not the case for $\mathfrak{p}(n)$. However, an analogous such series for $\mathfrak{p}(n,\chi_p)$ *could* be used to give more direct, analytic[1] proofs of periodic vanishings of some $\mathfrak{p}(n,\chi_p)$.

We are currently making progress on the following results, the first of which is to appear in an in-preparation manuscript by the author.

Theorem 3.2 (In preparation)**.** *One has*

$$\mathfrak{p}(n,\chi_{17}) = 0 \qquad \text{for all } n \equiv 17, 19, 25, 27 \pmod{34}.$$

Equivalently, one has $\mathfrak{p}(n,\chi_{17}) = 0$ precisely when n is odd and $1-24n$ is congruent to a quartic residue (mod 17)*. In addition, one has*

$$\mathfrak{p}(n,\chi_{17}^{\dagger}) = 0 \qquad \text{for all } n \equiv 11, 15, 29, 33 \pmod{34},$$

or, equivalently, for n odd and congruent to a quadratic-nonquartic residue (mod 17)*.*

Claim 3.3 (Proof in progress)**.** One further has

$$\begin{aligned}\mathfrak{p}(n,\chi_{17}) &= \mathfrak{p}(n,\chi_{17}^{\dagger}) &\qquad \text{for all } n \equiv 3, 7, 13, 31 \pmod{34},\\ \mathfrak{p}(n,\chi_{17}) &= -\mathfrak{p}(n,\chi_{17}^{\dagger}) &\qquad \text{for all } n \equiv 1, 9, 21, 23 \pmod{34}.\end{aligned}$$

Conjecture 3.4. The only odd primes for which $\mathfrak{p}(n,\chi_p)$ and $\mathfrak{p}(n,\chi_p^{\dagger})$ vanish on some arithmetic progressions $(\mathrm{mod}\, 2p)$, as seen above, are 5 and 17.

In particular, applying Rademacher's and Lehner's techniques [15, 17], one finds that for $p < 24$, one has

$$\begin{aligned}\mathfrak{p}(n,\chi_p) \sim &\sum_{\substack{k=1\\ 2\nmid k,\ p\nmid k}}^{\infty} \left\{\lambda_k \mathfrak{L}_k(n) + \lambda_{2k}\mathfrak{L}_{2k}(n)\right\} I_1\big(f(k,n)\big)/k,\\ &+ \sum_{\substack{k=1\\ 4\mid k,\ p\nmid k}}^{\infty} \left\{\lambda_k \mathfrak{L}_k(n)\right\} I_1\big(f(k,n)\big)/k + \sum_{\substack{k=1\\ 2\nmid k,\ p\mid k}}^{\infty} \left\{\mathfrak{L}_k^{+}(n)\right\} I_1\big(g(k,n)\big)/k,\end{aligned} \tag{3.5}$$

where the λ_k are related to elements of certain cyclotomic fields (see, e.g., [11, p. 10 ff.]), the function I_1 is the modified Bessel function of the first kind, and f and g are elementary functions. The quantity $\mathfrak{L}_k^{+}(n)$ here is a modified version of the sum (3.2), wherein the sum in (3.2) is changed to only sum over those $h \pmod{k}$ such that $(h,k) = 1$ and $\chi(h) = +1$.

Determination of the sums $\mathfrak{L}_k$ above requires detailed knowledge on congruences of the quantity $24k\Lambda(h,k)$ modulo $48k$. A number of lemmata on these congruences have been established by the author already; completion of a handful of further results will allow for Salié-like formulae for the Kloosterman sums

$$\mathfrak{L}_q(n,m) := \sideset{}{'}\sum_{h \,(\mathrm{mod}\, q)} \exp\left\{\pi i \Lambda(h,q) - 2\pi i (hn + \bar{2}\bar{h}m)/q\right\} \qquad (q = p^{\alpha}),$$

where $h\bar{h} \equiv 1 \pmod{q}$.

[1] Although the proofs of Theorem 2.1 given in [8] rely on intricate symbolic manipulations and well-known q-series identities, many of those used identities were historically discovered and proved using the theory of modular forms.

References

[1] G. E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[2] B. C. Berndt, *Reciprocity theorems for Dedekind sums and generalizations*, Advances in Math. **23** (1977), no. 3, 285–316.

[3] B. C. Berndt, A. Malik, and A. Zaharescu, *Partitions into kth powers of terms in an arithmetic progression*, Math. Z. **290** (2018), no. 3-4, 1277–1307.

[4] S. H. Chan, *Dissections of quotients of theta-functions*, Bull. Austral. Math. Soc. **69** (2004), no. 1, 19–24.

[5] S. Chern and D. Tang, *Vanishing coefficients in quotients of theta functions of modulus five*, Bull. Aust. Math. Soc. **102** (2020), no. 3, 387–398.

[6] T. Daniels, *Biasymptotics of the Möbius- and Liouville-signed partition numbers*, submitted; Oct. 2023 preprint arXiv:2310.10617, 38pp.

[7] T. Daniels, *Bounds on the Möbius-signed partition numbers*, Ramanujan J. **65** (2024), no. 1, 81–123.

[8] T. Daniels, *Vanishing coefficients in two q-series related to legendre-signed partitions*, Res. Number Theory **10** (2024), no. 4, Paper No. 81, 12 pp.

[9] T. Daniels, *Legendre-signed partition numbers*, J. Math. Anal. App. **542** (2025), no. 1, Paper No. 128717, 43 pp.

[10] M. Das, N. Robles, A. Zaharescu, and D. Zeindler, *Partitions into semiprimes*, arXiv preprint arXiv:2212.12489, 37pp.

[11] H. Davenport, *Multiplicative number theory*, Second, Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York-Berlin, 1980. Revised by Hugh L. Montgomery.

[12] P. Erdos, *On an elementary proof of some asymptotic formulas in the theory of partitions*, Ann. of Math. (2) **43** (1942), 437–450.

[13] G. H. Hardy and S. Ramanujan, *Asymptotic Formulae in Combinatory Analysis*, Proc. London Math. Soc. (2) **17** (1918), 75–115.

[14] M. D. Hirschhorn, *Two remarkable q-series expansions*, Ramanujan J. **49** (2019), no. 2, 451–463.

[15] J. Lehner, *A partition function connected with the modulus five*, Duke Math. J. **8** (1941), 631–655.

[16] J. Mc Laughlin, *New infinite q-product expansions with vanishing coefficients*, Ramanujan J. **55** (2021), no. 2, 733–760.

[17] H. Rademacher, *On the Partition Function $p(n)$*, Proc. London Math. Soc. (2) **43** (1937), no. 4, 241–254.

[18] D. D. Somashekara and M. B. Thulasi, *Results on vanishing coefficients in certain infinite q-series expansions*, Ramanujan J. **60** (2023), no. 2, 355–369.

[19] R. C. Vaughan, *On the number of partitions into primes*, Ramanujan J. **15** (2008), no. 1, 109–121.

[20] R. C. Vaughan, *Squares: additive questions and partitions*, Int. J. Number Theory **11** (2015), no. 5, 1367–1409.