When we did homogeneous linear ODEs, if we got a complex conjugate root pair $r = p \pm qi$, then originally we thought to use sol.

$$x(t) = c_1 e^{(p+qi)t} + c_2 e^{(p-qi)t}$$

as a general solution. However, we then expanded

$$\begin{aligned} x(t) &= c_1 e^{(p+qi)t} + c_2 e^{(p-qi)t} \\ &= c_1 e^{pt}(\cos(qt) + i\sin(qt)) \\ &\qquad + c_2 e^{pt}(\cos(qt) - i\sin(qt)) \\ &= \underbrace{(c_1+c_2)}_{=:A} e^{pt}\cos(qt) + \underbrace{i(c_1-c_2)}_{=:B} e^{pt}\sin(qt), \end{aligned}$$

and then later just jump to solution

$$x(t) = A e^{pt}\cos(qt) + B e^{pt}\sin(qt).$$

<u>Idea</u>: with conjugate root pair $p \pm qi$, we can get the same general solution by using the <u>real</u> and <u>imaginary</u> parts of the single "solution" $e^{(p+qi)t}$

complex number $\lambda = p + qi$

real part "$\mathrm{Re}[\lambda]$"

imaginary part "$\mathrm{Im}[\lambda]$"

So, going back to conjugate root pair $p \pm qi$, we can get the same general solution as before as follows:

Start with single solution $\underline{\tilde{x}(t) = e^{(p+qi)t}} = ?$

$$\Leftrightarrow = \underbrace{e^{pt}\cos(qt)}_{x_1(t) := Re[\tilde{x}(t)]} + i\underbrace{e^{pt}\sin(qt)}_{x_2(t) := Im[\tilde{x}(t)]}.$$

Then linear combinations

$$A\,x_1(t) + B\,x_2(t)$$
$$= A\,\underbrace{e^{pt}\cos(qt)} + B\,\underbrace{e^{pt}\sin(qt)}$$

produce the same general solutions as when we combined

$$c_1\,e^{(p+qi)t} + c_2\,e^{(p-qi)t}.$$

Starting point: Find solutions to the system $\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 3x_1 + 4x_2 \end{cases}$ $\quad \underline{x}' = \underline{A}\,\underline{x}$

$$A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$$

1) $|\underline{A} - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = 0$

$$\Rightarrow \lambda = 4 \pm 3i.$$

2) Eigenvector for $\lambda = 4+3i$:

Solve $(\underline{A} - (4+3i)I)\begin{bmatrix} a \\ b \end{bmatrix} = 0$

$$\begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$R_2 \to R_2 + (-i)R_1 \quad \begin{bmatrix} -3i & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$-3ia + -3b = 0$$

$$a = ib,$$

so solutions are $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ib \\ b \end{bmatrix} = b\begin{bmatrix} i \\ 1 \end{bmatrix}$,

so choose $\underline{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ as eigenvector for $\lambda = 4+3i$

Since the other eigenvalue is $4-3i = \overline{4+3i} = \bar{\lambda}$,

we know an eigenvector for $\bar{\lambda}$ will be $\overline{\underline{v}_1}$,

i.e. $\underline{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Instead of combining solutions

$$e^{(4+3i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} + e^{(4-3i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix},$$

we can get the same general solution

using the Re and Im parts of the single solution $\underline{e^{(4+3i)t}\begin{bmatrix} i \\ 1 \end{bmatrix}} =: \tilde{\underline{x}}(t)$

$$\tilde{\underline{x}}(t) = e^{4t}\left(\cos(3t) + i\sin(3t)\right)\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= e^{4t}\begin{bmatrix} i\cos(3t) - \sin(3t) \\ \cos(3t) + i\sin(3t) \end{bmatrix}$$

$$\tilde{\underline{x}}(t) = \underbrace{e^{4t}\begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix}}_{\underline{x}_1 := \mathrm{Re}[\tilde{\underline{x}}(t)]} + i\underbrace{e^{4t}\begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix}}_{\underline{x}_2 := \mathrm{Im}[\tilde{\underline{x}}(t)]}$$

General solution for $\underline{x}' = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\underline{x}$ will be

$$\underline{x}(t) = c_1\underline{x}_1(t) + c_2\underline{x}_2(t)$$

$$= c_1 e^{4t}\begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix}$$

$$= \begin{bmatrix} -c_1 e^{4t}\sin(3t) + c_2 e^{4t}\cos(3t) \\ c_1 e^{4t}\cos(3t) + c_2 e^{4t}\sin(3t) \end{bmatrix}$$

$$\underline{x}_1(t) = \begin{bmatrix} -e^{4t}\sin(3t) \\ e^{4t}\cos(3t) \end{bmatrix}, \quad \underline{x}_2(t) = \begin{bmatrix} e^{4t}\cos(3t) \\ e^{4t}\sin(3t) \end{bmatrix}$$

are the real-valued solutions for the system.

$\lambda = 4 \pm 3i$

$\uparrow$

$\underline{e^{4t}}$