Ex: $\underline{A} = \begin{bmatrix} -11 & 0 & -4 \\ -1 & -9 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, $|\underline{A}-\lambda \underline{I}| = 0 \Rightarrow \ldots$

$\ldots \Rightarrow \lambda = -9, -9, -9$

Solve $\left(\underline{A} - (-9)\underline{I}\right)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$R_1 \overset{\text{swap}}{\longleftrightarrow} R_2$ $\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ -2 & 0 & -4 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$

$\left.\begin{array}{l} R_1 \to (-1)R_1 \\ R_2 \to R_2 + 2R_3 \end{array}\right\}$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$

$R_2 \overset{\text{swap}}{\longleftrightarrow} R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$R_2 \to R_2 - R_1$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$b$ free, $a, c$ leading $\left\{ \begin{array}{l} a + c = 0 \Rightarrow a = 0 \\ c = 0 \end{array}\right.$

so solutions for $\left(\underline{A} - (-9)\underline{I}\right)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$ are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

only one basis vector, pick $b=1$, $\underline{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Because $\lambda = -9$ has multiplicity 3, but only one lin. indep eigenvector (the $\underline{v_1}$ we just found),

$\lambda = -9$ has defect $d = 3 - 1 = 2$.

Use (method 2) to make a "chain" of length 3, namely just want $\underline{v_1}, \underline{v_2}, \underline{v_3} \neq 0$ such that

$$\begin{cases} (\underline{A} - \lambda I)^3 \underline{v_3} = 0 \\ \underline{v_2} := (\underline{A} - \lambda I)\underline{v_3} & (\text{needs to be } \neq 0) \\ \underline{v_1} := (\underline{A} - \lambda I)\underline{v_2} & (\quad " \quad " \quad " \quad " \quad) \end{cases}$$

Using $\underline{A}$ from example and $\lambda = -9$, we find that

$$(\underline{A} - (-9)I)^3 \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

so $a, b, c$ free. Pick $a = 1$, $b = 0$, $c = 0$, $\underline{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then

$$\underline{v_2} = (\underline{A} - (-9)I)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -1 \\ +1 \end{bmatrix} \quad (\neq 0)$$

$$\underline{v_1} = (\underline{A} - (-9)I)\begin{bmatrix} -2 \\ -1 \\ +1 \end{bmatrix}$$

$$= \cdots = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(which happens to be the same $\underline{v_1}$ we found earlier, but it isn't always like this)

What if we picked $a=0$, $b=1$, $c=0$ and made $\underline{v_3} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

Then
$$\underline{v_2} = (\underline{A} - (-9)I)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ +1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

but we said $\underline{v_2}$ can't be $0$. ($\underline{v_1}$ would also be $=0$ here, which also can't happen)

Hence we shouldn't use the choice $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

* for <u>this</u> problem.

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ might work in other problems.

So we have $\underline{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} -2 \\ -1 \\ +1 \end{bmatrix}$, $\underline{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Solutions
$$\underline{x_1}(t) := e^{\lambda t}\underline{v_1} = e^{-9t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\underline{x_2}(t) := e^{\lambda t}(t\underline{v_1} + \underline{v_2})$$
$$= e^{-9t}\left(t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ +1 \end{bmatrix}\right)$$
$$\underline{x_3}(t) = e^{\lambda t}\left(\frac{t^2}{2!}\underline{v_1} + t\,\underline{v_2} + \underline{v_3}\right)$$
$$= e^{-9t}\left(\frac{t^2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

General solution is
$$\underline{x}(t) = c_1\,\underline{x_1}(t) + c_2\,\underline{x_2}(t) + c_3\,\underline{x_3}(t).$$

If we needed $\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4} \neq 0$ (in a different problem)

such that $(\underline{A} - \lambda I)^4 \underline{v_4} = 0$

and chain
$$\begin{cases} (\underline{A} - \lambda I)\underline{v_4} = \underline{v_3} \\ (\underline{A} - \lambda I)\underline{v_3} = \underline{v_2} \\ (\underline{A} - \lambda I)\underline{v_2} = \underline{v_1} \\ (\underline{A} - \lambda I)\underline{v_1} = 0 \end{cases},$$

then we'd have everything like before, but also add-in

$$\underline{x_4}(t) = e^{\lambda t}\left(\frac{t^3}{3!}\underline{v_1} + \frac{t^2}{2!}\underline{v_2} + t\underline{v_3} + \underline{v_4}\right)$$

However, these are lengthy and don't show up in problems too much.