Homework 9.

3) Consider a population $P(t)$ of rabbits satisfying the Logistic Eqn

$$\frac{dP}{dt} = aP - bP^2$$

where $B(t) = aP(t)$ is the time rate at which births occur (e.g., births/month), and $D(t) = bP(t)^2$ is the time rate at which deaths occur (e.g., deaths/month). If the initial population is 240 rabbits, and there are 9 births/month and 12 deaths/month occuring at $t=0$, how many months does it take for $P(t)$ to reach 105% of the limiting population M?

Idea: usually we work with the Logistic equation in the form

$$\frac{dP}{dt} = kP(M-P) \qquad (k, M > 0), \quad (\text{"Form 3"}).$$

In order to deal with this problem, we should change

$$P' = aP - bP^2$$

into our usual form. So we write

$$P' = aP - bP^2 = bP\left(\frac{a}{b} - P\right),$$

indicating that our equation is

$$P' = kP(M-P) \quad \text{with} \quad \begin{matrix} k = b \\ M = a/b \end{matrix}.$$

Once we have k and M, we can go to the solution

$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-kMt}}.$$

So let's find k and M; to do this we need a and b.

We have $B(t) = aP(t)$ is the births/month rate that is occuring at time t. At $t=0$, $B(0) = 9$, and $P(0) = 240$. But then

$$B(0) = aP(0) \implies 9 = a(240) \implies \underline{a = \frac{9}{240}}.$$

Similarly, since $D(t) = bP(t)^2$ and $D(0) = 12$, we get

$$D(0) = bP(0)^2 \implies 12 = b(240)^2 \implies b = \frac{12}{(240)^2}.$$

Thus $a = \dfrac{9}{240}$ and $b = \dfrac{12}{(240)^2}$, meaning

$$k = b = \frac{12}{(240)^2} = \frac{12}{(12\cdot 20)^2} = \frac{12}{(12)^2 400} = \frac{1}{12\cdot 400} = \frac{1}{4800}$$

and $M = \dfrac{a}{b} = \dfrac{9/240}{12/(240)^2} = \dfrac{9}{12}\cdot 240 = \dfrac{3}{4}\cdot 240 = 180.$

So the limiting population $M = 180$, and we want $P(t) = (1.05)M$

Before turning to the calculator we can simplify a bit more:

Want $(1.05)M = P(t)$

$$\Rightarrow \quad (1.05)M = \frac{P_0 M}{P_0 + (M-P_0)e^{-kMt}}$$

$$\Rightarrow \quad (1.05) = \frac{P_0}{P_0 + (M-P_0)e^{-kMt}}$$

$$\Rightarrow \quad P_0 + (M-P_0)e^{-kMt} = \frac{P_0}{1.05}$$

Now it is fine to plug in the values for k, M, and P_0, and solve for t.