Second order ODEs involve x, y, y', y''.

A 2^{nd} order ODE is reducible if either x or y is missing.

Reducible 2^{nd} order ODEs can be solved using substitution.

Case: y is missing, ex: $xy'' + 2y' = 6x$.

In this case, let

$$v = y', \quad \text{so that} \quad y'' = (v)' = v'.$$

Ex: Solve the ODE $xy'' + 2y' = 6x$.

Since y is missing, let

$$v = y', \quad \text{so that} \quad y'' = v'.$$

Then the ODE is

$$xv' + 2v = 6x$$

$$v' + \frac{2}{x}v = 6.$$

(Notice we have changed ("reduced") a 2^{nd} order ODE into a 1^{st} order)

This is a linear ODE ($v' + P(x)v = Q(x)$), and, solving it, we find that

$$v(x) = 2x + \frac{c_1}{x^2}.$$

But then $y' = 2x + \frac{c_1}{x^2} \Rightarrow \boxed{y = x^2 - \frac{c_1}{x} + c_2}$

Note we have two different constants. This is because we solved a 2^{nd} order ODE.

Case: x is missing, ex: $y'' + 9y = 0$.

In this case, again let

$$v = y'.$$

This time though, write

$$y'' = v' = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{dv}{dy} \cdot y' = \frac{dv}{dy} \cdot v,$$

in total: $y'' = v \cdot \frac{dv}{dy}$.

Ex: Solve the ODE $y'' + 9y = 0$. assume x, y, y' are >0 where needed.

Since x is missing, let $v = y'$ so that $y'' = v \cdot \frac{dv}{dy}$, and the ODE becomes

$$v \cdot \frac{dv}{dy} + 9y = 0.$$

$$v \frac{dv}{dy} + 9y = 0$$

$$\Rightarrow \quad v\,dv = -9y\,dy$$

$$\Rightarrow \quad \frac{1}{2}v^2 = -\frac{9}{2}y^2 + c$$

(Trick) $\Rightarrow \quad v^2 = -9y^2 + C \quad \leftarrow$ ("2c" $\to$ "c")

↳ The fact that $C = v^2 + 9y^2$ means $C \geq 0$, which means that $k := \sqrt{C}$ is well defined, so

$$y' = v = \sqrt{C - 9y^2}$$

$$y' = \sqrt{k^2 - 9y^2}$$

$$y' = 3\sqrt{(k/3)^2 - y^2}$$

$$y' = 3\sqrt{(k_2)^2 - y^2} \qquad (k_2 := k/3)$$

This says

$$\frac{dy}{dx} = 3\sqrt{k_2^2 - y^2} \qquad \text{(separable)}$$

$$\rightarrow \underbrace{\int \frac{dy}{\sqrt{k_2^2 - y^2}}}_{\sin^{-1}(y/k_2)} = \int 3\,dx$$

good to remember that $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$!

$$\sin^{-1}(y/k_2) = 3x + \tilde{C}$$

$$\Rightarrow \quad y = k_2 \sin(3x + \tilde{C}).$$

Since $k_2 = \frac{\sqrt{C}}{3}$ is just some unknown constant, we can just wrap it up as "k," and since $\tilde{C}$ is another unknown constant, and since we aren't using the label "C" anymore (since it got wrapped up into the label "k", we can just write "c" in place of "$\tilde{C}$".

Alltogether then, our solution is

$$\boxed{y = k \sin(3x + C)}$$

Again, it is expected to have two unknown constants because we started with a second order equation.