Picking v

We saw three "types" of substitution problems in class.

1) If $\frac{dy}{dx} = F(ax+by+c)$, where $F(\ )$ is just some expression and/or function, then use

$$v := ax+by+c.$$

For example, we saw that for

$$\frac{dy}{dx} = (x+y+3)^2,$$

we used

$$v = x+y+3. \qquad (\Rightarrow y = v-x-3)$$

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2) If $\frac{dy}{dx} = F(y/x)$, then use

$$v := y/x.$$

For example, in class we changed

$$2xy\,y' = 4x^2+3y^2 \longrightarrow y' = 2\left(\frac{x}{y}\right) + \frac{3}{2}\left(\frac{y}{x}\right) \qquad \left(= \frac{2}{(y/x)} + \frac{3}{2}\cdot(y/x)\right)$$

$$\longrightarrow y' = \frac{2}{v} + \frac{3}{2}\cdot v \qquad 1$$

Equations of the form $\frac{dy}{dx} = F(y/x)$ are "Homogeneous Eqns," which means they remain unchanged if you change both $\begin{cases} x \to cx \\ y \to cy \end{cases}$. For example, above, the equation is

$$2xy\frac{dy}{dx} = 4x^2+3y^2$$

If we change $\begin{matrix} x \to cx \\ y \to cy \end{matrix}$, then, first, $\frac{dy}{dx} \longrightarrow \frac{d(cy)}{d(cx)} = \frac{c}{c}\cdot\frac{dy}{dx} = \frac{dy}{dx}$ (constants pass through derivatives), so $\frac{dy}{dx}$ is unchanged. Then

$$2(cx)(cy)\frac{dy}{dx} = 4(cx)^2+3(cy)^2$$

$$2\cancel{c^2}xy\frac{dy}{dx} = 4\cancel{c^2}x^2+3\cancel{c^2}y^2$$

$$2xy\frac{dy}{dx} = 4x^2+3y^2,$$

which is what we started with. Then we change to 1

We might've guessed this would happen since all terms had the same degree, which is 2.

x^2 is degree 2, y^2 is degree 2, xy is degree $1+1=2$,

so the change $\begin{cases} x \to cx \\ y \to cy \end{cases}$ resulted in c^2 on both sides of the equation, which cancelled.

→ whenever we have a homogeneous equation (the degree may be different than 2), that is when we have/can change our equation into

$$\frac{dy}{dx} = F(y/x),$$

and so this is when we use

$$v = y/x \qquad (\Rightarrow \ y = xv).$$

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3) Finally we saw Bernoulli Equations, which are

$$y' + P(x)y = Q(x)y^{1-n}.$$

If $n=0$ or $n=1$, we saw this makes a linear equation.

When $n \neq 0$, $n \neq 1$, we use

$$v = y^{1-n} \qquad (\Rightarrow \ y = v^{\frac{1}{1-n}})$$

These 3 types are the most common substitution equations we'll deal with. However... there could be problems which are different, in which you are given a specific choice of $v = \dots$ to use. In that case you simply use that v and follow the method.