General solutions & unknown constants

Starting with

$$\frac{dy}{dx} = f(x),$$

integrate both sides with respect to x

$$\rightarrow \quad \int \frac{dy}{dx}\, dx = \int f(x)\, dx.$$

"Cancel" $\rightarrow \quad \int 1\, dy^{*} = \int f(x)\, dx$

$$\Rightarrow \quad y + C_1 = \int f(x)\, dx,$$

* Since $\int 1\, dx = x + C$, it is sensible that $\int 1\, dy = y + C$

where C_1 is an unknown constant. Since we want a formula for $y(x)$, we write

$$y(x) = \int f(x)\, dx - C_1.$$

Now, c_1 is an unknown constant, and therefore $-c_1$ is also another unknown constant. As we progress, we will often have multiple unknown constants being combined, which gets tedious. To help this, we can combine unknown constants that are multiplied or added with each other. (more on that below)

For starters, above we said $-c_1$ is an unknown constant. Instead of writing

$$y(x) = \int f(x)\, dx - c_1,$$

we "wrap" the minus sign "into" the unknown constant c_1, and package this as "C," and thus write

$$y(x) = \int f(x)\, dx + C. \qquad ①$$

Consider now general solutions to the ODE

$$y' = 2x \qquad \text{like in class.}$$

From ① we can say

$$y(x) = \underbrace{\int 2x\, dx}_{x^2 + \tilde{c}} + C,$$

$$\Rightarrow \quad y(x) = x^2 + \tilde{C} + C$$

where $\tilde{c}$ is an unknown constant, and so is C. But since $\tilde{c}, c$ are both unknown constants, the quantity $\tilde{c}+c$ is again an unknown constant. Rather than package them up as, for example, $\tilde{\tilde{c}} := c + \tilde{c}$, we just stick with the simple label for the entire package, "c".

Thus the general solution is written

$$y(x) = x^2 + c.$$

It's important to keep in mind that we only do this with unknown constants. For example, in constant acceleration we saw that $x(t) = c_2 + c_1 t + \frac{1}{2}at^2$, (though of course we know that $c_1 = v_0$ and $c_2 = x_0$).

c_2 and c_1 would be unknown constants, but $c_1 t$ is not constant since t is the variable, so we don't combine $c_2 + c_1 t$ "=" c.

What happens if we have, or leave "extra" constants while solving an IVP?

For example,

$$\left\{ y' = \frac{1}{\sqrt{x}}, \quad y(4) = 7. \right\}$$

If we write

$$y = \int \frac{1}{\sqrt{x}}\,dx + c_1, \qquad \text{(like from ① on previous page)}$$

we might then write

$$y = 2\sqrt{x} + c_2 + c_1 \qquad ②$$

as our general solution. Then we need

$$7 = y(4) = 2\sqrt{4} + c_2 + c_1$$

$$\Rightarrow \quad 7 = 4 + c_2 + c_1$$

so

$$c_1 + c_2 = 3. \qquad ③$$

We don't know c_1 and c_2 individually, but our concern is just ② and its particular solution. In ② we see that $\boxed{c_1 + c_2}$ is what we need, and so, using ③ in ② we conclude

$$y(x) = 2\sqrt{x} + 3.$$

Since only $c_1 + c_2$ mattered in the end, we could preemptively package up $C := c_1 + c_2$, so that

$$y = 2\sqrt{x} + C$$

is the general solution, then computed

$$7 = y(4) = 2\sqrt{4} + C \quad \Rightarrow \quad C = 3$$

and thus arrived to the same particular solution

$$y(x) = 2\sqrt{x} + 3 .$$

This "packaging" applies to general constant quantities too. If c_1, c_2, c_3 were all unknown <u>constants</u>, then something like $2c_1 - \frac{\sqrt{c_2}}{c_3^2+1}$ is again just some constant,

so then it is much easier to write something like

$$y(x) = \tan^{-1}\left(\frac{x}{4}\right) + \boxed{C}$$

instead of

$$y(x) = \tan^{-1}\left(\frac{x}{4}\right) + \boxed{2c_1 - \frac{\sqrt{c_2}}{c_3^2+1}}$$