HOMEWORK #2 - MA 504

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Chapter 1, problem 6. Fix $b > 1$.
(a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.
(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational numbers.
(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.
(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Solution.
(a) By theorem 1.21 there is one and only one positive real y such that $y^n = b^m$, and write $y = (b^m)^{1/n}$. Similarly $\exists!$ (This symbol means "there exists one and only one") $z \in \mathbb{R}$ such that $z^q = b^p$, and write $z = (b^p)^{1/q}$. We want to show that $z = y$. We have

$$(y^n)^p = (b^m)^p = \underbrace{\underbrace{(b \cdots b)}_{m \text{ times}} \cdots \underbrace{(b \cdots b)}_{m \text{ times}}}_{p \text{ times}} = \underbrace{\underbrace{(b \cdots b)}_{p \text{ times}} \cdots \underbrace{(b \cdots b)}_{p \text{ times}}}_{m \text{ times}} = (b^p)^m = (z^q)^m.$$

Hence $y^{np} = z^{qm} = x$. By hypothesis $np = qm = d$. It follows by theorem 1.21 that $\exists! w \in \mathbb{R}$ s.t. $w^d = x$. Therefore, by uniqueness, $y = w = z$. Q.E.D.
(b) Let $r = m/n, s = a/c \in \mathbb{Q}, c, n > 0$. We have then $r + s = \dfrac{mc + na}{nc}$. We want to show that

$$b^{r+s} = (b^{mc+na})^{\frac{1}{nc}} = (b^m)^{1/n}(b^a)^{1/c} = b^r b^s.$$

As we showed in the previous item, by the uniqueness part of theorem 1.21, it sufficies to show that

$$(b^{r+s})^{nc} = (b^r b^s)^{nc}.$$

Similarly as we showed in the previous item, one can show $(b^{r+s})^{nc} = b^{mc+na} = b^{mc} b^{na}$. We have by associativity that

$$(b^r b^s)^{nc} = \underbrace{b^r b^s \cdots b^r b^s}_{nc \text{ times}} = (\underbrace{b^r \cdots b^r}_{nc \text{ times}})(\underbrace{b^s \cdots b^s}_{nc \text{ times}}) = (b^r)^{nc}(b^s)^{nc}.$$

One can also show by similar methods that $(b^r)^{nc} = [(b^r)^n]^c = [(b^r)^c]^n$, and $(b^s)^{ma} = [(b^s)^m]^a = [(b^s)^a]^m$.
By definition
$$(b^r)^n = [(b^m)^{1/n}]^n = b^m.$$
Similarly $(b^s)^c = b^a$. Then
$$(b^r b^s)^{nc} = [(b^r)^n]^c[(b^s)^c]^n = (b^m)^c(b^a)^n = b^{mc}b^{na} = b^{mc+na} = (b^{r+s})^{nc}. \quad \text{Q.E.D.}$$
(c) We want to show that
$$b^r = \sup B(r)$$
when r is rational. First of all, we have to show that b^r is an upper bound of $B(r)$. We have if $t \in \mathbb{Q}, t \leq r$, then $r = t + r - t$ and $r - t \geq 0$. So, by the previous item
$$b^r = b^t b^{r-t}.$$
Now since $b > 1$ and $r - t \geq 0$, $b^{r-t} \geq 1$. Indeed, write $r - t = k/j \in \mathbb{Q}, k \geq 0, j > 0$, then $b^{r-t} = (b^k)^{1/j}$. So if $b^{r-t} < 1$, then
$$(b^{r-t})^j = [(b^k)^{1/j}]^j = b^k < 1,$$
but this is a contradiction with $b > 1, k \geq 0, k \in \mathbb{Z}$. Hence $b^{r-t} \geq 1$ and then $b^r = b^{r-t}b^t \geq b^t$. Since $t \leq r$ is arbitrary, we have that
$$b^r \geq \sup B(r).$$
Now since $b^r \in B(r)$, we have
$$b^r \leq \sup B(r),$$
so $b^r = \sup B(r)$. We then define
$$b^x = \sup B(x), x \in \mathbb{R}.$$
(d) We want to show that $b^{x+y} = b^x b^y$, for any real x and y. We have
$$b^{x+y} = \sup B(x+y) = \sup\{b^t : t \in \mathbb{Q}, t \leq x + y\}.$$
If $r, s \in \mathbb{Q}$ are such that $r \leq x$, $s \leq y$, then $r+s \in \mathbb{Q}$ and $r+s \leq x+y$. Then $b^{r+s} \in B(x+y)$, so
$$b^r b^s = b^{r+s} \leq \sup B(x+y).$$
Since $r \leq x$ is arbitrary, we can take the supremum over b^r on the left hand side and get
$$(\sup B(x))\, b^s \leq \sup B(x+y).$$
Similarly, taking the supremum over b^s, we have
$$(\sup B(x))\,(\sup B(y)) \leq \sup B(x+y),$$
ie, $b^x b^y \leq b^{x+y}$.
Conversely, if $t \in \mathbb{Q}, t < x + y$, then $t - y < x$. By the discution given in section 1.22, we see that we can take $r \in \mathbb{Q}$ such that $t - y \leq r \leq x$, so $t - r \in \mathbb{Q}$ and $t - r \leq t - (t - y) = y$. Hence
$$b^t = b^r b^{t-r} \leq (\sup B(x))\,(\sup B(y))\,.$$

Since $t < x + y$ is arbitrary, we have

$$\sup B(x+y) \le (\sup B(x))\,(\sup B(y))\,,$$

and then

$$b^{x+y} = \sup B(x+y) = (\sup B(x))\,(\sup B(y)) = b^x b^y. \quad \text{Q.E.D.}$$

Chapter 1, problem 15. Under what conditions does equality hold in the Schwartz inequality?

Solution.
The Schwartz inequality (theorem 1.35) says that if $a_1, ..., a_n$ and $b_1, ..., b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

Let $x = (a_1, ..., a_n), y = (b_1, ..., b_n)$. We have,

$$|x-y|^2 = (x-y)\cdot(x-y) = x\cdot x - 2x\cdot y + y\cdot y = |x|^2 - 2x\cdot y + |y|^2.$$

So if the equality holds in the Schwartz inequality, we have $x \cdot y = |x||y|$, hence

$$|x-y|^2 = |x|^2 - 2|x||y| + |y|^2 = (|x| - |y|)^2.$$

Now we see that we may assume without loss of generality that $|x| = |y| = 1$. Indeed, if $x = 0$ or $y = 0$, then $x \cdot y = 0 = |x||y|$, otherwise consider the vectors $x/|x|$ and $y/|y|$, note that $\left(\frac{x}{|x|}\right) \cdot \left(\frac{y}{|y|}\right) = \frac{1}{|x||y|}(x \cdot y) = \frac{1}{|x||y|}(|x||y|) = \left|\frac{x}{|x|}\right|\left|\frac{y}{|y|}\right| = 1$. Then we have $|x-y|^2 = (|x|-|y|)^2 = 0$, so $x = y$.
Therefore the equality holds in the Schwartz inequality if and only if $x = 0$ or $y = 0$ or $\frac{x}{|x|} = \frac{y}{|y|}$, i.e., $x = \lambda y, \lambda \in \mathbb{R}$, x is parallel to y.

Chapter 1, problem 16. Suppose $k \ge 3, x, y \in \mathbb{R}^k, |x-y| = d > 0$, and $r > 0$. Prove:
(a) If $2r > d$, there are infinitely many $z \in \mathbb{R}^k$ such that

$$|z-x| = |z-y| = r.$$

(b) If $2r = d$, there exists one such z.
(c) If $2r < d$, there exists no such z.
How must these statements be modified if k is 2 or 1?

Solution.
(a) First of all, let us have a geometric view of $|x-y|$. $|x-y|$ is the distance of x to y, so the points $z \in \mathbb{R}^k$ such that $|z-x| = r$ give the sphere of center x and radius r, call it $S_r(x)$. Hence $|z-x| = |z-y| = r$ gives the intersection of the spheres with centers x and

y, respectively, and radius r.
Let $\omega \in \mathbb{R}^k$, $|\omega| = 1$, and $\omega \perp (x-y)$, ie, $\omega \cdot (x-y) = 0$. Let

$$z = \frac{x+y}{2} + \left(\sqrt{r^2 - \frac{d^2}{4}}\right)\omega.$$

Note that since $k \geq 3$, there are infinitely many $\omega \in \mathbb{R}^k$, $|\omega| = 1$, with $\omega \perp (x-y)$. Now since $\omega \perp (x-y)$

$$|z-x|^2 = \left|\frac{x-y}{2}\right|^2 + \left(r^2 - \frac{d^2}{4}\right)|\omega|^2 = \frac{d^2}{4} + r^2 - \frac{d^2}{4} = r^2. \tag{0.1}$$

Similarly one shows that $|z-y| = r$.
Therefore there are infinitely z s.t. $|z-x| = |z-y| = r$. Note that we strongly use that $2r > d$, otherwise $r^2 - \dfrac{d^2}{4} \leq 0$. Also, the equation above, (0.1), caractherizes all such points z.
(b) By theorem 1.37 (f), triangle inequality,

$$d = |x-y| \leq |z-x| + |z-y| = 2r.$$

Now note that $2r = d$ implies that the equality holds, but the equality holds if and only if $z = \dfrac{x+y}{2}$. Therefore there is exactly one z s.t. $|z-x| = |z-y| = d/2$, namely $z = (x+y)/2$.
(c) If $2r < d$, then clearly by the triangle inequality, one can see that there is no z such that $|z-x| = |z-y| = r$.
Note that in the case $k = 2$, if $2r > d$ then there exists exactly two points, say $z_1, z_2 \in \mathbb{R}^k$, such that $|z_j - x| = |z_j - y| = r, j = 1, 2$. If $k = 2$ and $2r = d$, then the conclusion is the same as before. If $k = 1$, then if $2r \neq d$, there is no such z, if $2r = d$, then there is exactly one such z.

Chapter 1, problem 17. Prove that

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$$

if $x, y \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Solution.
Previously we showed

$$|x-y| = |x|^2 - 2x \cdot y + |y|^2,$$
$$|x+y| = |x|^2 + 2x \cdot y + |y|^2.$$

Then clearly we have what is asked for, the so called Parallelogram law, which states that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.

Chapter 1, problem 18. If $k \geq 2$ and $x \in \mathbb{R}^k$, prove that there exists $y \in \mathbb{R}^k$ such that $y \neq 0$, but $x \cdot y = 0$.

Solution.
Let $x = (x_1, ..., x_k) \in \mathbb{R}^k$. Assume that $x \neq 0$, otherwise $x \cdot y = 0$ for all $y \in \mathbb{R}^k$. Hence there exists at least one $x_j, j = 1, ..., k$, such that $x_j \neq 0$. Now given any $k-1$ real numbers $y_1, y_2, ..., y_{j-1}, y_{j+1}, ..., y_k$, with at least one different from zero, consider

$$y_j = -\frac{1}{x_j}(x_1y_1 + x_2y_2 + ... + x_{j-1}y_{j-1} + x_{j+1}y_{j+1} + ... + x_ky_k).$$

By the choice of the y_l's, $l = 1, ..., k$, $y = (y_1, ..., y_k) \in \mathbb{R}^k$, $y \neq 0$ and $x \cdot y = 0$. We say that y belongs to the $k-1$ dimensional hyperplane $x^{\perp} = \{z \in \mathbb{R}^k : x \cdot z = 0\}$.

Chapter 2, problem 4. Is the set of irrational real numbers countable?

Solution.
The answer is no. Indeed if we assume that the set of irrational real numbers, say $\mathbb{R} \setminus \mathbb{Q}$, is countable, then the sets $[0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$ and $[0,1] \cap \mathbb{Q}$ would be countable, since are subsets of countable sets, namely $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{Q}$. Then

$$[0,1] = ([0,1] \cap (\mathbb{R} \setminus \mathbb{Q})) \cup ([0,1] \cap \mathbb{Q})$$

would be also countable, since it would be a union of two countable sets. But this is a contradiction with the corollary of theorem 2.43.

Chapter 2, problem 8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Solution.
The answer is SIM! (SIM = YES in portuguese) Indeed If $E \subset \mathbb{R}^2$ is an open set, then every point $p \in E$ is an interior point of E, ie, there exists a neighborhood N of p such that $N \subset E$. Now given any neighborhood G of p, by theorem 2.24 $G \cap N$ is open, so there exists a neighborhood H of p such that $H \subset G \cap N \subset N \subset E$. So for any $q \in H, q \neq p$, $q \in G \cap E$, hence, since G is arbitrary, p is a limit point of E. Note that this proof works for if E is a subset of a general topological space, in particular for metric spaces.
The same property does not hold for general closed sets. For instance (0,1) is a point in the closed set $(0,1), (2,2), (3,1) \subset \mathbb{R}^2$ that is not a limit point.

Chapter 2, problem 9. Let E° denote the set of all interior points of E.
(a) Prove that E° is always open.
(b) Prove that E is open if and only if $E^{\circ} = E$.
(c) If $G \subset E$ and G is open, prove that $G \subset E^{\circ}$.

Solution.
(a) If $E^{\circ} =$, then it is clearly open, so assume $E^{\circ} \neq$. Let $p \in E^{\circ}$. Then p is an interior point of E, ie, there exists a neighborhood of p, say N, such that $N \subset E$. Since N is open, by definition, $N = N^{\circ}$ and clearly $N^{\circ} \subset E^{\circ}$. Therefore $N = N^{\circ} \subset E^{\circ}$ and since $p \in E^{\circ}$ is arbitrary, E° is open.

(b) Clearly if E is open, then by definition $E = E^\circ$. Conversely if $E = E^\circ$, then every point of E is interior, so E is open by definition.
(c) Let $G \subset E$ and G open. Then we have

$$G = G^\circ \subset E^\circ.$$