MA 261

Quiz 10 12 aprilo 2016

Instructions: Write your name and section number on your quiz. Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

Problem 1. (5 points) Is the vector field $\mathbf{F} = \left\langle \frac{x}{y}, \frac{y}{z}, \frac{z}{x} \right\rangle$ a conservative vector field? (Give evidence.)

Solution. Recall that $\mathbf{F}$ is conservative if its domain is all of $\mathbb{R}^3$, has continuous partial derivatives, and if $\operatorname{curl}\mathbf{F} = 0$. But $\mathbf{F}(0,0,0)$ is undefined, hence $\mathbf{F}$ is not conservative. If we computed the curl, we would find

$$\begin{aligned}
\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{y} & \frac{y}{z} & \frac{z}{x} \end{vmatrix} &= \left[\frac{\partial}{\partial y}\left(\frac{z}{x}\right) - \frac{\partial}{\partial z}\left(\frac{y}{z}\right)\right]\mathbf{i} \\
&- \left[\frac{\partial}{\partial x}\left(\frac{z}{x}\right) - \frac{\partial}{\partial z}\left(\frac{x}{y}\right)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\left(\frac{y}{z}\right) - \frac{\partial}{\partial y}\left(\frac{x}{y}\right)\right]\mathbf{k} \\
&= \left[0 - \left(-\frac{y}{z^2}\right)\right]\mathbf{i} - \left[-\frac{z}{x^2} - 0\right]\mathbf{j} + \left[0 - \left(-\frac{x}{y^2}\right)\right]\mathbf{k} \\
&= \left\langle \frac{y}{z^2}, \frac{z}{x^2}, \frac{x}{y^2} \right\rangle \neq 0,
\end{aligned}$$

so we could also conclude from this that $\mathbf{F}$ is not conservative. ☺

Problem 2. (5 points) Compute $\operatorname{div}\mathbf{F}$, where $\mathbf{F} = \langle e^x \sin y, e^y \sin z, e^z \sin x \rangle$.

Solution.

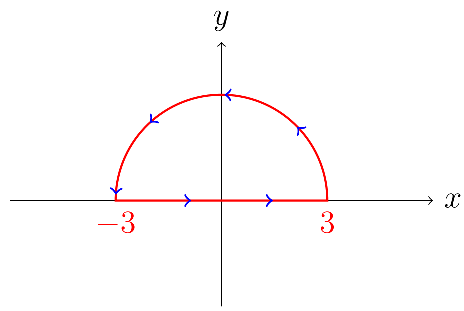
$$\begin{aligned}
\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle e^x \sin y, e^y \sin z, e^z \sin x \rangle \\
&= \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^y \sin z) + \frac{\partial}{\partial z}(e^z \sin x) \\
&= e^x \sin y + e^y \sin z + e^z \sin x
\end{aligned}$$

☺

Problem 3. (5 points) Evaluate

$$\int_C y\,\mathrm{d}x + x\,\mathrm{d}y,$$

where C is the upper half-circle of radius 3 centered at the origin oriented counterclockwise and the line segment from $x = -3$ to $x = 3$ (pictured below).



Solution. Notice that the curve on which we wish to integrate is simple closed, and positively oriented, so we may employ Green's Theorem. Now denote by D the region enclosed by C. Then

$$\int_C y\,\mathrm{d}x + x\,\mathrm{d}y = \iint_D \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}y\right)\mathrm{d}A = \iint_D (1-1)\,\mathrm{d}A = \iint_D 0\,\mathrm{d}A = 0. \quad ☺$$

Problem 4. (0 points) What is your favorite thing about Purdue and/or the greater Lafayette metropolitan area?