MA 261

Quiz 11 19 abril 2016

Instructions: Write your name and section number on your quiz. Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

Problem 1. (10 points) Find a parametric representation for the sphere

$$x^2 + y^2 + z^2 = 16$$

that lies between the planes $z = -2$ and $z = 2$.

Solution. Recall the standard parametrization of the sphere of radius R:

$$\mathbf{r}(\theta, \varphi) = (R \sin\varphi \cos\theta, R \sin\varphi \sin\theta, R \cos\varphi),$$

where $0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi$. In this problem, we have $R = 4$. But since we want $-2 \leq z \leq 2$, the only thing left to do is figure out how we want to restrict φ. Now

$$\begin{aligned} z = 4\cos\varphi &= 2 \\ \Leftrightarrow \cos\varphi &= \frac{1}{2} \\ \Leftrightarrow \varphi &= \frac{\pi}{3}, \end{aligned}$$

and by symmetry $z = -2$ implies that we have $\varphi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. ☺

Problem 2. (10 points) Evaluate

$$\iint\limits_S xz \, \mathrm{d}S,$$

where S is the part of the plane $2x + 2y + z = 4$ that lies in the first octant.

Solution. S is the surface

$$z = 4 - 2x - 2y$$

over the region

$$D = \{(x, y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq 2 - x\}$$

Recall formula 4 in Stewart tells us that

$$\iint\limits_S f(x, y, z) \, \mathrm{d}S = \iint\limits_D f\left(x, y, g(x, y)\right) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, \mathrm{d}A,$$

so we have

$$\begin{aligned}
\iint\limits_S xz\,\mathrm{d}S &= \iint\limits_D x(4-2x-2y)\sqrt{(-2)^2+(-2)^2+1}\,\mathrm{d}A \\
&= 3\int_0^2\int_0^{2-x}(4x-2x^2-2xy)\,\mathrm{d}y\,\mathrm{d}x \\
&= 3\int_0^2\left[4xy-2x^2y-xy^2\right]_{y=0}^{y=2-x}\,\mathrm{d}x \\
&= 3\int_0^2\left[4x(2-x)-2x^2(2-x)-x(2-x)^2\right]\mathrm{d}x \\
&= 3\int_0^2\left(x^3-4x^2+4x\right)\mathrm{d}x = 3\left[\frac{1}{4}x^4-\frac{4}{3}x^3+2x^2\right]_0^2 \\
&= 3(4-\frac{32}{3}+8) \\
&= 4
\end{aligned}$$

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Problem 3. (0 points) Are you doing anything fun for the summer?