MA 261

Quiz 3 2 febbraio 2016

Instructions: Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

Problem 1. (5 points) [From 13.2, Example 4]
Show that if $\left|\mathbf{r}(t)\right| = c$ (a constant) then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.
(Hint: $\left|\mathbf{r}(t)\right|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$.)

Solution. Observe that $\left|\mathbf{r}(t)\right| = c \Rightarrow \left|\mathbf{r}(t)\right|^2 = c^2$ Using the hint, we see that $\mathbf{r}(t) \cdot \mathbf{r}(t) = c^2$. Thus if we take derivative of both sides, we find

$$\begin{aligned}
\big(\mathbf{r}(t) \cdot \mathbf{r}(t)\big)' &= \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) && \text{by Leibniz rule} \\
&= 2\mathbf{r}(t) \cdot \mathbf{r}'(t) && \text{since } \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \\
&= 0 && \text{since } c \text{ is a constant}
\end{aligned}$$

Hence $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, and $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$. ☺

Note that $\left|\mathbf{r}(t)\right| = c$ does not imply $\mathbf{r}(t) = c$ (and thus does not imply that $\mathbf{r}'(t) = 0$). The counter example I gave in class was to take $\mathbf{r}(t) = (\cos t, \sin t)$, the unit circle traversed counter-clockwise. Then $\left|\mathbf{r}(t)\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$, a constant, but $\mathbf{r}(t)$ is clearly not constant, and $\mathbf{r}'(t) = (-\sin t, \cos t)$ is not the zero vector. But when we take $\mathbf{r}(t) \cdot \mathbf{r}'(t) = (\cos t, \sin t) \cdot (-\sin t, \cos t) = -\sin t \cos t + \sin t \cos t = 0$.

Problem 2. (5 points) [Taken from Exam 1, SP15, #6]
Find the length of the curve

$$\mathbf{r}(t) = \ln(t)\mathbf{i} + \sqrt{2}t\mathbf{j} + \frac{t^2}{2}\mathbf{k}$$

for $1 \leq t \leq 2$.

Solution. We compute the arclength by integrating the speed over the given interval. First note that $\mathbf{r}'(t) = \frac{1}{t}\mathbf{i} + \sqrt{2}\mathbf{j} + t\mathbf{k}$, so that

$$\left|\mathbf{r}'(t)\right| = \sqrt{\frac{1}{t^2} + 2 + t^2} = \sqrt{\left(\frac{1}{t} + t\right)^2} = \frac{1}{t} + t$$

(since $0 < 1 \le t$). Now,

$$\begin{aligned} L &= \int_1^2 \left|\mathbf{r}'(t)\right| dt = \int_1^2 \left(\frac{1}{t} + t\right) dt \\ &= \ln t + \frac{t^2}{2}\bigg|_1^2 = (\ln 2 + 2) - \left(\ln 1 + \frac{1}{2}\right) = \ln 2 + \frac{3}{2}. \end{aligned}$$

☺

Problem 3. (5 points)[Taken from 13.4.9]
If the position of a particle is given by

$$\mathbf{r}(t) = \left\langle t^2 + t, t^2 - t, t^3 \right\rangle,$$

find its speed when $t = 0$.

Solution. The speed is given by $\left|\mathbf{v}(t)\right| = \left|\mathbf{r}'(t)\right|$. Computing the velocity, we find $\mathbf{v}(t) = \left\langle 2t+1, 2t-1, 3t^2 \right\rangle$, so $\left|\mathbf{v}(t)\right| = \sqrt{(2t+1)^2 + (2t-1)^2 + (3t^2)^2}$. Now at $t = 0$, $\left|\mathbf{v}(t)\right| = \sqrt{1+1+0} = \sqrt{2}$. ☺