Name: Solutions

ID number: ______________________

Low: 40
Median: 68
Mean: 66.5
High: 98

A: 7 75 (8)
B: 65 - 75 (10)
C: 55 - 64 (12)
D: 45 - 54 (2)
F: < 45 (4)

Instructions:

1. This is a one-hour exam.
2. There are 11 problems on this exam.
3. No books, notes, or calculators are allowed.
4. Please turn off your cell phone.
5. Circle one and only one choice for each multiple-choice problem. No partial credit will be given for multiple-choice problems.
6. Show all relevant work on non-multiple-choice problems. Partial credit will be given for steps leading to the correct solutions. Write your final answer in the box provided.
7. You may use a writing utensil, your own brain and the paper provided in this exam. Use of any other persons or resources will be considered cheating and will be reported to the Office of the Dean of Students.

I agree to abide by the instructions above:

Signature: ______________________

Page	Score	Points Possible
2		10
3		10
4		22
5		12
6		12
7		12
8		12
9		10
Total		100

1. (5 points) Given that $\mathbf{u} \cdot \mathbf{v} = 2\sqrt{3}$ and $\mathbf{u} \times \mathbf{v} = \langle 2, 4, 4 \rangle$, what is the measure of the angle between $\mathbf{u}$ and $\mathbf{v}$?

A. 0

B. $\frac{\pi}{2}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{4}$

E. $\frac{\pi}{6}$

$|u \times v| = |u||v| \sin\theta$

$u \cdot v = |u||v| \cos\theta$

$\Rightarrow \frac{|u \times v|}{u \cdot v} = \tan\theta$

$|u \times v| = 6$

$\tan\theta = \frac{6}{2\sqrt{3}} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

2. (5 points) Describe the level curves of $f(x, y) = \ln\sqrt{4x^2 + y^2}$.

A. Circles

B. Hyperbolas

C. Ellipses

D. Logarithmic curves

E. Parabolas

$\ln(4x^2 + y^2)^{\frac{1}{2}} = k$

$\ln(4x^2 + y^2) = 2k$

$4x^2 + y^2 = e^{2k}$

3. (5 points) If $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$, describe the set of all points such that $|\mathbf{r} - \mathbf{r}_0| = 1$.

A. The cylinder in the direction of $\mathbf{r}_0$ with radius 1.

B. The sphere with center $\mathbf{r}_0$ and radius 1.

C. The set of all unit vectors in $\mathbb{R}^3$.

D. A plane containing $\mathbf{r}_0$.

E. The set of all vectors parallel to $\mathbf{r}_0$.

4. (5 points) Suppose $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$ are all differentiable functions. Using the tables of values below, determine $\frac{\partial z}{\partial s}$ at the point $(s, t) = (0, 0)$.

(x,y)	$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$
$(0,0)$	2	0	2
$(1,2)$	3	-3	4
$(4,-5)$	3	4	2

(s,t)	$g(s,t)$	$h(s,t)$	$g_s(s,t)$	$h_s(s,t)$
$(0,0)$	4	-5	1	2
$(1,2)$	5	2	0	0
$(4,-5)$	1	2	0	4

A. 0

B. 2

C. 4

D. 8

E. There is insufficient information to compute $\frac{\partial z}{\partial s}$ at the given point.

$(s,t) = (0,0) \Rightarrow (x,y) = (4,-5)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= f_x(4,-5)\,g_s(0,0) + f_y(4,-5)\,h_s(0,0)$$

$$= (4)(1) + (2)(2)$$

$$= 8$$

5. (10 points) Show that $\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t)\cdot\mathbf{r}'(t)$. [Hint: $|\mathbf{r}(t)|^2 = \mathbf{r}(t)\cdot\mathbf{r}(t)$.]

$$|\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$$

$$\frac{d}{dt}|\vec{r}(t)|^2 = \frac{d}{dt}\vec{r}(t)\cdot\vec{r}(t)$$

$$2|\vec{r}(t)|\frac{d}{dt}|\vec{r}(t)| = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t)$$

$$= 2\vec{r}(t)\cdot\vec{r}\,'(t)$$

$$\Rightarrow \frac{d}{dt}|\vec{r}(t)| = \frac{1}{|\vec{r}(t)|}\vec{r}(t)\cdot\vec{r}\,'(t)$$

6. (12 points) Reparametrize the curve $\mathbf{r}(t) = \langle 3\cos t, 2t, 3\sin t\rangle$ with respect to arc length starting from the point $(3, 0, 0)$ in the direction of increasing t.

$$\vec{r}\,'(t) = \langle -3\sin t, 2, 3\cos t\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{9\sin^2 t + 4 + 9\cos^2 t} = \sqrt{13}$$

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{13}\,du = \sqrt{13}\,t.$$

$$\Rightarrow t = s/\sqrt{13}$$

$$\boxed{\vec{r}(t(s)) = \left\langle 3\cos\left(\frac{s}{\sqrt{13}}\right), 2\frac{s}{\sqrt{13}}, 3\sin\left(\frac{s}{\sqrt{13}}\right)\right\rangle}$$

7. (12 points) A particle starts at the origin with initial velocity **j**. Its acceleration is

$$\mathbf{a}(t) = 4t^3\mathbf{i} - e^{-t}\mathbf{j} + \cos(2t)\mathbf{k}.$$

Find its position function. You may leave your answer in **i**, **j**, **k** notation, a vector equation or parametric equations.

$\vec{v}(t) = \langle t^4, e^{-t}, \frac{1}{2}\sin 2t \rangle + \vec{C}$

$\langle 0,1,0\rangle = \vec{v}(0) = \langle 0,1,0\rangle + \vec{C} \implies \vec{C} = \vec{0}$

$\vec{r}(t) = \langle \frac{1}{5}t^5, -e^{-t}, -\frac{1}{4}\cos 2t \rangle + \vec{D}$

$\langle 0,0,0\rangle = \vec{r}(0) = \langle 0, -1, -\frac{1}{4}\rangle + \vec{D} \implies \vec{D} = \langle 0, 1, \frac{1}{4}\rangle$

$$\boxed{\vec{r}(t) = \langle \tfrac{1}{5}t^5, 1-e^{-t}, \tfrac{1}{4} - \tfrac{1}{4}\cos 2t \rangle}$$

8. (12 points) Consider the curve $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 4, \sqrt{2}t - 1, \ln t \right\rangle$. You may use (without showing) that it has $\mathbf{r}'(t) = \left\langle t, \sqrt{2}, \frac{1}{t} \right\rangle$ and $|\mathbf{r}'(t)| = t + \frac{1}{t}$. Find

(a) the vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{t+\frac{1}{t}} \langle t, \sqrt{2}, \tfrac{1}{t} \rangle = \left\langle \frac{t^2}{t^2+1}, \frac{\sqrt{2}\,t}{t^2+1}, \frac{1}{t^2+1} \right\rangle$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{t^2+1}{\sqrt{2}} \left\langle \frac{2t}{(t^2+1)^2}, \frac{\sqrt{2}(1-t^2)}{(t^2+1)^2}, \frac{-2t}{(t^2+1)^2} \right\rangle$$

$$\vec{T}'(t) = \left\langle \frac{2t}{(t^2+1)^2}, \frac{\sqrt{2}(1-t^2)}{(t^2+1)^2}, \frac{-2t}{(t^2+1)^2} \right\rangle$$

$$|\vec{T}'(t)| = \sqrt{\frac{4t^2 + 2(1-t^2)^2 + 4t^2}{(t^2+1)^4}} = \frac{\sqrt{2}}{t^2+1}$$

$$\boxed{\begin{array}{l} \vec{T}(t) = \left\langle \frac{t^2}{t^2+1}, \frac{\sqrt{2}\,t}{t^2+1}, \frac{1}{t^2+1} \right\rangle \\ \vec{N}(t) = \left\langle \frac{\sqrt{2}\,t}{t^2+1}, \frac{1-t^2}{t^2+1}, \frac{-\sqrt{2}\,t}{t^2+1} \right\rangle \end{array}}$$

(b) its curvature $\kappa(t)$.

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

$$\kappa(t) = \frac{\sqrt{2}/(t^2+1)}{t + \frac{1}{t}} = \frac{\sqrt{2}\,t}{(t^2+1)^2}$$

$$\boxed{\kappa(t) = \frac{\sqrt{2}\,t}{(t^2+1)^2}}$$

9. (12 points) Find the limit, if it exists, or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^5}$

Along $y = mx$: $\displaystyle\frac{x^2(mx)^2}{x^4+(mx)^5} = \frac{m^2x^4}{x^4(1+m^5x)} = \frac{m^2}{1+m^5x} \longrightarrow \frac{m^2}{1+m^5}$

as $x \to 0$. This limit depends on m,

so limit does not exist.

$\boxed{\text{DNE}}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x+y}$

$\displaystyle\frac{x^3+y^3}{x+y} = \frac{\cancel{(x+y)}(x^2-xy+y^2)}{\cancel{x+y}} = x^2-xy+y^2 \longrightarrow 0$

by continuity.

$\boxed{0}$

10. (12 points) Consider the function $f(x, y) = \ln(x^2 - y^3)$.

(a) What is the domain of f?

ln requires $x^2 - y^3 > 0$

$\Leftrightarrow \quad x^2 > y^3$

$$\boxed{\{(x,y) \mid x^2 > y^3\}}$$

(b) Compute f_{xy}.

$$f_x = \frac{2x}{x^2 - y^3} = 2x(x^2 - y^3)^{-1}$$

$$f_{xy} = 2x \cdot (-1)(x^2 - y^3)^{-2} \cdot (-3y^2)$$

$$= \frac{6xy^2}{(x^2 - y^3)^2}$$

$$\boxed{\frac{6xy^2}{(x^2-y^3)^2}}$$

11. (10 points) Find an equation of the tangent plane to the surface

$$z = xe^{xy}$$

at the point $(2, 0, 2)$.

$$z - 2 = f_x(2,0)(x-2) + f_y(2,0)(y-0)$$

$f_x = xye^{xy} + e^{xy} \Rightarrow f_x(2,0) = 1$

$f_y = x^2 e^{xy} \Rightarrow f_y(2,0) = 4$

$$z - 2 = 1(x-2) + 4y$$

$$\boxed{z = x + 4y}$$