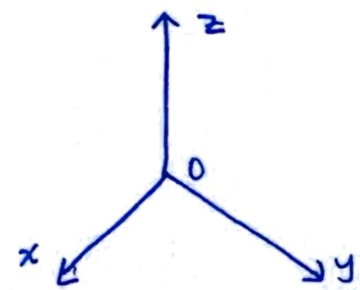
Section 12.1 3-D Coord. Systems

Right-hand rule: Point fingers in positive x direction, curl toward positive y direction, then your thumb points in positive z direction.

We denote the origin by O.



The first octant is the part where x, y, z all positive.

Points are now ordered triples (a,b,c), or $P(a,b,c)$ to denote the point P with coordinates (a,b,c).

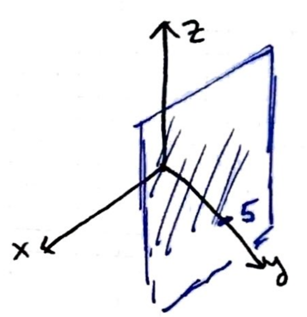
This space is called $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) : x,y,z \in \mathbb{R}\}$.
Recall that $\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$ is a plane.

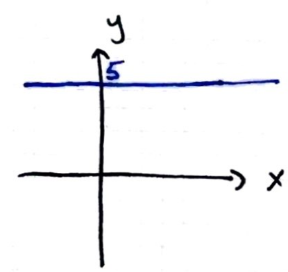
Equations of surfaces

Note: An equation in x and y in $\mathbb{R}^2$ is a curve, but an equation in x and y in $\mathbb{R}^3$ is a surface.

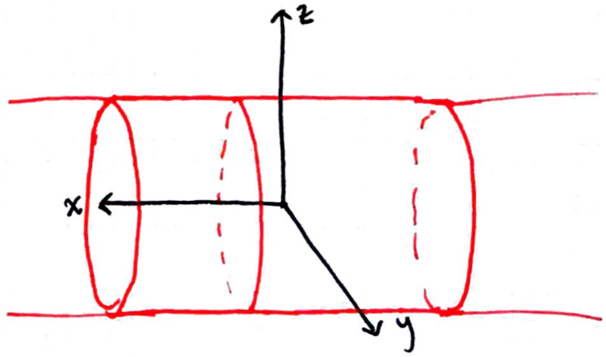
Example 1

(a) $y = 5$ in $\mathbb{R}^3$ is a plane
in $\mathbb{R}^2$ is a line

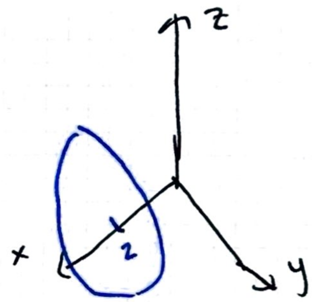




(b) What does the equation $y^2 + z^2 = 1$ represent in $\mathbb{R}^3$?



(c) What if we restrict to $x = 2$?



circle of radius 1 at $x = 2$, center $(2, 0, 0)$.

<u>Distance formula</u> The distance $|P_1 P_2|$ between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

<u>Why</u>? Comes from two applications of Pythagorean Theorem

<u>Consequence</u>: Eqn of sphere with center $C(h, k, l)$, radius r is

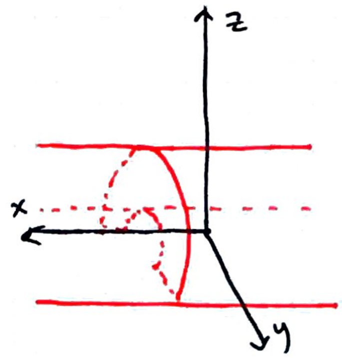
$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

If the center is the origin, this simplifies to

$$x^2 + y^2 + z^2 = r^2.$$

Example 2 (a) What region is represented by

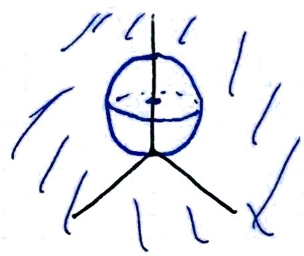
$$1 \le y^2 + z^2 \le 4, \quad z \ge 0?$$



(b) $x^2 + y^2 + z^2 > 2z$

Complete the square: $x^2 + y^2 + (z-1)^2 > 1$

This is everything outside of the sphere of radius 1 with center $(0,0,1)$

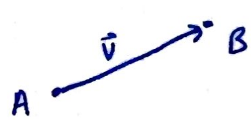


Remark often necessary to complete the square to identify circle.

Section 12.2 Vectors

A point denotes a position in space, while a vector denotes an incremental change.
In $\mathbb{R}^2$ or $\mathbb{R}^3$, can be thought of as a quantity with magnitude and direction. Denoted by $\mathbf{v}$ or $\vec{v}$.

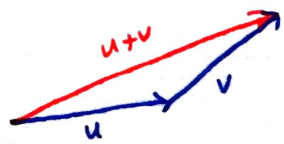
A displacement vector gives the displacement from A to B



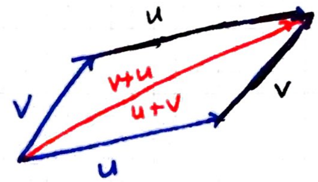
$\vec{v} = \overrightarrow{AB}$

Two vectors with same magnitude and direction are equal.
The zero vector, denoted by 0 or $\vec{0}$, has length 0, no direction.

<u>Vector addition</u>: For vectors $\vec{u}, \vec{v}$, the sum $\vec{u}+\vec{v}$ is given by



The Triangle Law

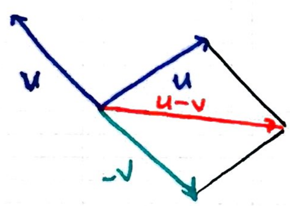


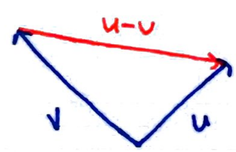
The Parallelogram Law

The Parallelogram Law states that vector addition is commutative, that is, $u+v = v+u$.

<u>Scalar multiplication</u> If $c \in \mathbb{R}$ (called a scalar) and $\vec{v}$ a vector, then $c\vec{v}$ has length $|c|$ times the length of $\vec{v}$ whose direction is the same as $\vec{v}$ if $c>0$, opposite if $c<0$, and if $c=0$ or $\vec{v}=0$, then $c\vec{v}=0$.

The difference $\vec{u}-\vec{v} = \vec{u}+(-\vec{v})$

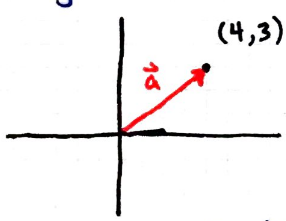




Can be done with Parallelogram Law or Triangle Law

<u>Components</u>

We introduce a coordinate system to work with vectors algebraically.



The coordinates of the vector are called its components, write $\vec{a} = \langle 4,3 \rangle$ to differentiate from points.

This is a <u>representation</u> of the vector $\langle 4,3 \rangle$. Any vector that has initial point (a,b) and terminal point $(a+4, b+3)$ will be equivalent. The particular representation starting from the origin is called the <u>position vector</u>.

Given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\vec{a}$ with representation $\vec{AB}$ is

$$\vec{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Example 3 Find the vector represented by the directed line segment with initial point $A(1,2,1)$ and terminal point $B(-1, 0, 3)$.

Solution $\vec{a} = \langle -1-1, 0-2, 3-1 \rangle = \langle -2, -2, 2 \rangle$.

The magnitude or length of the vector $\vec{v}$, denoted $|\vec{v}|$ or $\|\vec{v}\|$, is the length of any of the representations of $\vec{v}$. Use the distance formula from the origin. In $\mathbb{R}^3$,

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}, \quad \text{where } \vec{a} = \langle a_1, a_2, a_3 \rangle.$$

Vector operations algebraically

In $\mathbb{R}^2$: $\vec{a} = \langle a_1, a_2 \rangle$, $b = \langle b_1, b_2 \rangle$.

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c \cdot \hat{a} = \langle ca_1, ca_2 \rangle$$

Sim for $\mathbb{R}^3$.

Example 4 Given $\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle 2, 2, 2 \rangle$, find $|\vec{a}|$, $\vec{a} + \vec{b}$ $\vec{a} - 3\vec{b}$ $4\vec{a} + 2\vec{b}$.

$$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\vec{a} + \vec{b} = \langle 1+2, 2+2, 3+2 \rangle = \langle 3, 4, 5 \rangle$$

$$\vec{a} - 3\vec{b} = \langle 1-6, 2-6, 3-6 \rangle = \langle -5, -4, -3 \rangle$$

$$4\vec{a} + 2\vec{b} = \langle 4, 8, 12 \rangle + \langle 4, 4, 4 \rangle = \langle 8, 12, 16 \rangle$$

Properties of vectors $\vec{u}, \vec{v}, \vec{w}$ any vectors. a, b scalars.

1) $u + v = v + u$
2) $u + (v + w) = (u + v) + w$
3) $u + 0 = u$
4) $u + (-u) = 0$
5) $a(u + v) = au + av$
6) $(a+b)u = au + bu$
7) $(ab)u = a(bu)$
8) $1 \cdot u = u$

Easily verified algebraically or geometrically

In $\mathbb{R}^3$ the standard basis vectors are

$$\hat{i} = \langle 1,0,0\rangle, \quad \hat{j} = \langle 0,1,0\rangle, \quad \hat{k} = \langle 0,0,1\rangle.$$

In $\mathbb{R}^2$,

$$\hat{i} = \langle 1,0\rangle, \quad \hat{j} = \langle 0,1\rangle$$

These vectors have length 1 and point in the positive direction of the axes.

We can write any vector in terms of $\hat{i}, \hat{j}, \hat{k}$.

e.g. $\langle 3,-1,2\rangle = 3\hat{i} - \hat{j} + 2\hat{k}$

A unit vector is a vector whose length is 1. If $\vec{a} \neq 0$, then the unit vector in the same direction as $\vec{a}$ is

$$\vec{u} = \frac{1}{|\vec{a}|} \cdot \vec{a} = \frac{\vec{a}}{|\vec{a}|}.$$

Section 12.3 The Dot Product.

Given $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$ then the dot product of $\vec{a}$ and $\vec{b}$, denoted $\vec{a}\cdot\vec{b}$ is

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Also salled scalar product or inner product.

Example 5

$$\langle 3,2\rangle \cdot \langle 1,4\rangle = 3\cdot 1 + 2\cdot 4 = 11$$

$$(-\hat{i} + \hat{j} + 2\hat{k})\cdot(-\hat{j} + 3\hat{k}) = (-1)(0) + (1)(-1) + (2)(3) = 5$$

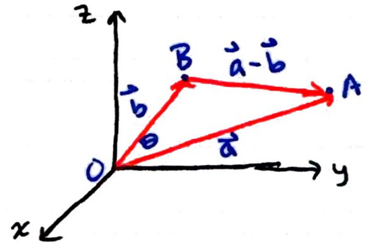
Properties of the dot product $\vec{u}, \vec{v}, \vec{w}$ vectors, a scalar.

1) $\vec{u}\cdot\vec{u} = |\vec{u}|^2$ 2) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
3) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ 4) $(a\vec{u})\cdot\vec{v} = \vec{u}\cdot(a\vec{v})$
5) $\vec{0}\cdot\vec{u} = 0$.

Easily verified by definition.

Theorem If θ is the angle between vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$



proof Law of Cosines: $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$. (*)

Using properties of dot product:

$$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = a\cdot a - a\cdot b - b\cdot a - b\cdot b$$

$$= |a|^2 - 2a\cdot b + |b|^2$$

Thus (*) gives

$$|a|^2 - 2a\cdot b + |b|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$$

$$-2a\cdot b = -2|a||b|\cos\theta$$

$$\boxed{a\cdot b = |a||b|\cos\theta}$$

$\boxtimes$

Example 6 Find the angle btwn $\vec{a} = \langle 1, 0, 0\rangle$ and $b = \langle 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{1\cdot 2} \Rightarrow \theta = \frac{\pi}{3}$$

Two vectors are orthogonal or perpendicular if the angle between them is $\theta = \pi/2$. Using the theorem, we get

Two vectors $\vec{a}, \vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b} = 0$..

Section 12.4 The cross product

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ are vectors in $\mathbb{R}^3$, then the cross product of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

Note Result is a vector. Only defined for $\mathbb{R}^3$

Easier to remember using determinants.

Determinant of $2 * 2$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For 3×3, use expansion by minors

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

So we can write

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

Or

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad (*)$$

Remark First row of $(*)$ consists of vectors, so this is a memory tool rather than actually computing the determinant.

Example 7 $\langle 1, 0, -3 \rangle \times \langle 2, -4, 6 \rangle$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -3 \\ 2 & -4 & 6 \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ -4 & 6 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & -3 \\ 2 & 6 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 0 \\ 2 & -4 \end{vmatrix} \hat{k}$$

$$= 12\hat{\imath} - (6+6)\hat{\jmath} + (-4)\hat{k}$$

$$= \langle 12, -12, -4 \rangle$$

Theorem The vector $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a},\vec{b}$.

Proof We need to verify $(a\times b)\cdot a = 0$, $(a\times b)\cdot b = 0$. We show the former.

$$(\vec{a}\times\vec{b})\cdot\vec{a} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1a_2b_3 - a_1a_3b_2 - a_1a_2b_3 + a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_1$$

$$= 0. \quad //$$

Remark Cross product is given by right hand rule.

Theorem If θ is the angle between $\vec{a},\vec{b}$ $(0\le\theta\le\pi)$, then

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta.$$

Proof
$$|\vec{a}\times\vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$\begin{aligned} &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 \\ &\quad + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 \\ &\quad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \end{aligned}$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$= |a|^2|b|^2(1 - \cos^2\theta)$$

$$= |a|^2|b|^2\sin^2\theta$$

Taking square roots gives result since $\sqrt{\sin^2\theta} = \sin\theta$ for $0\le\theta\le\pi$. //

Corollary Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel iff $\vec{a}\times\vec{b} = 0$.

Proof Parallel iff $\theta = 0$ or π. Either way, $\sin\theta = 0$.
So $|a\times b| = 0 \Rightarrow a\times b = 0$. //

Geometric interpretation

Area $= |a|(|b| \sin\theta) = |a \times b|$

Example 8 Find a nonzero vector orthogonal to the plane passing through $P(1,0,1)$, $Q(-2,1,3)$, $R(4,2,5)$

Solution The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to $\vec{PQ}$ and $\vec{PR}$
$\Rightarrow$ orthogonal through the plane defined by $\vec{PQ}, \vec{PR}$.

$\vec{PQ} = \langle -3, 1, 2 \rangle$
$\vec{PR} = \langle 3, 2, 4 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \hat{i} - \begin{vmatrix} -3 & 2 \\ 3 & 4 \end{vmatrix} \hat{j} + \begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix} k$$

$$= (4-4)\hat{i} - (-12-6)\hat{j} + (-6-3)k$$

$$= 0\hat{i} + 18\hat{j} - 9\hat{k}$$

Example 9 Find area of triangle w/vertices P, Q, R.

Example 8 : $\vec{PQ} \times \vec{PR} = \langle 0, 18, -9 \rangle$. So area of parallelogram is $|\vec{PQ} \times \vec{PR}| = \sqrt{18^2 + 9^2} = 9\sqrt{5}$

Area of $PQR = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{9}{2}\sqrt{5}$.

Properties of Cross product

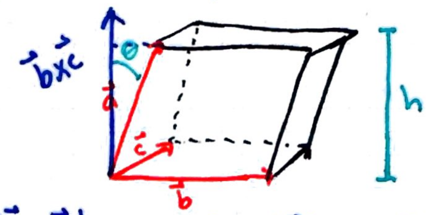
If $\vec{a}, \vec{b}, \vec{c}$ are vectors, α a scalar, then

1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2) $(\alpha \vec{a}) \times \vec{b} = \alpha(\vec{a} \times \vec{b}) = \vec{a} \times (\alpha \vec{b})$
3) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$

Triple products $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called scalar triple product

Can verify: $a\cdot(b\times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Geometric meaning: volume of parallelopiped spanned by a, b, c.



Area of base: $A = |\vec{b}\times\vec{c}|$, $h = |\vec{a}| \cos\theta$

$V = Ah = |\vec{b}\times\vec{c}|\,|\vec{a}|\cos\theta = |a\cdot(b\times c)|.$