14.7 Max & min values (cont'd)

Recall If f has local extremum and f_x, f_y exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$.

SDT f_x, f_y continuous on disk centered at (a,b), $f_x(a,b) = f_y(a,b) = 0$.

$$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

(a) If $D > 0$, $f_{xx}(a,b) > 0$, then $f(a,b)$ local min
(b) If $D > 0$, $f_{xx}(a,b) < 0$, then $f(a,b)$ local max
(c) If $D < 0$, $f(a,b)$ is a saddle point.

A closed set is one which contains all its boundary points. A boundary point of D is a point (a,b) such that every disk with center (a,b) contains points of D.

A bounded set in $\mathbb{R}^2$ is one that is contained in some disk.

Eg. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$ is closed and bounded. The boundary of D is $x^2 + y^2 = 1$.

Extreme Value Theorem for two variables If f is continuous on a closed and bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Remark The absolute extrema on a closed and bounded set D, are either critical points or points on the boundary of D.

To find absolute extrema on D:

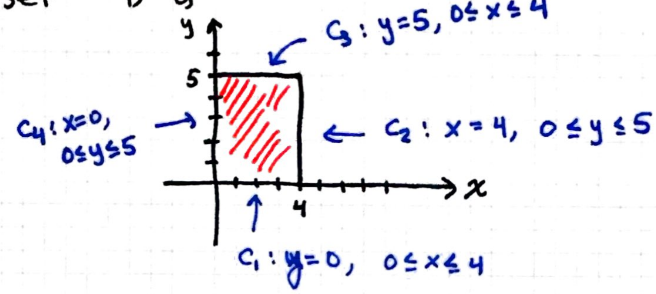
1) Find values of f at critical points in D
2) Find extreme values of f on boundary of D.
3) The largest of the values from 1) and 2) is the absolute max, and the smallest is the absolute min.

Note It is not necessary to apply the SDT in this case.

Example 1 Find the absolute max/min values of f on D, where

$$f(x,y) = 4x + 6y - x^2 - y^2, \quad D = \{(x,y) \mid 0 \le x \le 4,\ 0 \le y \le 5\}.$$

Solution The set D is



Finding critical points: $f_x(x,y) = 4 - 2x$, $f_y(x,y) = 6 - 2y$
$f_x = 0 \Rightarrow x = 2$, $f_y = 0 \Rightarrow y = 3$, so $(2,3)$ is the only critical point. Now for the boundary:

On C_1: $y=0 \Rightarrow f(x,y) = f(x,0) = 4x - x^2$, $0 \le x \le 4$. The critical numbers are $x = 0, 2, 4$. (The end points plus where $g'(x) = 0$, where $g(x) = f(x,0)$.)
Now $f(0,0) = 0$, $f(2,0) = 4$, $f(4,0) = 0$.

On C_2: $x=4 \Rightarrow f(x,y) = f(4,y) = 4(4) + 6y - (4)^2 - y^2 = 6y - y^2$, $0 \le y \le 5$.
The critical numbers here are $y = 0, 3, 5$. Now $f(4,0) = 0$, $f(4,3) = 9$ and $f(4,5) = 5$.

On C_3: $y=5 \Rightarrow f(x,y) = f(x,5) = 4x + 6(5) - x^2 - (5)^2 = 5 + 4x - x^2$, $0 \le x \le 4$.
This yields critical numbers same as C_1: 0, 2, 4. Now we check
$f(0,5) = 5$, $f(2,5) = 9$, $f(4,5) = 5$.

On C_4: $x=0 \Rightarrow f(x,y) = f(0,y) = 6y - y^2$, $0 \le y \le 5$. This yields critical numbers $y = 0, 3, 5$ like C_2. Now $f(0,0) = 0$, $f(0,3) = 9$, $f(0,5) = 5$.

The critical point $(2,3)$ is in D and $f(2,3) = 13$. We conclude the absolute max is 13, occurs at $(2,3)$; the absolute min is 0 and occurs at $(0,0)$ and $(4,0)$.

Example 2 Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4,2,0)$.

Solution Recall the distance function from the point $(4,2,0)$ is

$$d(x,y,z) = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$$

Note that d is nonnegative, so it is minimized precisely when d^2 is. So it suffices to minimize

$$D = d^2 = (x-4)^2 + (y-2)^2 + z^2.$$

Moreover, we have a constraint. Namely we want to restrict ourselves to the points (x,y,z) that lie on the cone $z^2 = x^2+y^2$. So we can replace z^2 in D by x^2+y^2:

$$\begin{aligned} D &= (x-4)^2 + (y-2)^2 + x^2+y^2 \\ &= x^2-8x+16 + y^2-4y+4 + x^2+y^2 \\ &= 2x^2+2y^2-8x-4y+20. \end{aligned}$$

Now $D_x = 4x-8 \overset{\text{set}}{=} 0 \Rightarrow x=2$, $D_y = 4y-4 \overset{\text{set}}{=} 0 \Rightarrow y=1$.
So the only critical point is $(2,1)$. We should confirm this is a minimum by SDT: $D_{xx}=4$, $D_{yy}=4$, $D_{xy}=0$, so yes, it is.

Finally, $x=2, y=1 \Rightarrow z^2 = 4+1 = 5 \Rightarrow z = \pm\sqrt{5}$, so the closest points are $(2,1,\pm\sqrt{5})$.

14.8 Lagrange multipliers

Set up: Given $f(x,y)$, we want to maximize/minimize $f(x,y)$ subject to some constraint $g(x,y)=k$.
Notice: $g(x,y)=k$ is a level curve of a function g, in other words,

We are looking for extreme values of $f(x,y)$ when (x,y) is restricted to lie on $g(x,y)=k$. So we are looking for the largest (or smallest) level curve of f that intersects the level curve $g(x,y)=k$. This means that the tangent vectors of the level curves must be parallel, which happens when $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$.

The number λ is called a Lagrange multiplier.

Method of Lagrange multipliers

To find max and min values of $f(x,y,z)$ subject to constraint $g(x,y,z)=k$ [assuming that these values exist and $\nabla g \neq 0$ on the surface $g(x,y,z)=k$]:

(a) Find all values of x, y, z and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and $$g(x,y,z) = k.$$

(b) Evaluate f at all points (x,y,z) in (a). The largest is the max; the smallest is the min.

Example 3 Use Lagrange multipliers to find the max and min values subject to the given constraint.

$$f(x,y) = xe^y \; ; \qquad x^2 + y^2 = 2.$$

Solution

$$f_x = e^y = \lambda \cdot 2x = \lambda g_x \qquad (1)$$
$$f_y = xe^y = \lambda \cdot 2y = \lambda g_y \qquad (2)$$

Note that $x \neq 0$ because e^y is never 0. So we can solve (1) for λ: $\lambda = \frac{1}{2x} e^y$. Substituting into (2):

$$xe^y = \tfrac{1}{2x} e^y \cdot 2y$$
$$\Leftrightarrow \quad 2x^2 = 2y$$
$$\Leftrightarrow \quad y = x^2$$

Using the constraint, $2 = x^2 + y^2 = x^2 + x^4$. This gives

$$x^4 + x^2 - 2 = 0$$
$$(x^2+2)(x^2-1) = 0.$$

So $x = \pm 1$. Since $y = x^2$, $x = \pm 1 \Rightarrow y = 1$. This gives us two points $(1,1)$ and $(-1,1)$. Now $f(1,1) = e$ and $f(-1,1) = -e$. So the minimum is $-e$ and the maximum is e.

Example 4 Same question for

$$f(x,y,z) = x^2 + y^2 + z^2 \; ; \qquad x^4 + y^4 + z^4 = 1.$$

Solution

$$f_x = 2x = \lambda g_x = \lambda \cdot 4x^3 \qquad (1)$$
$$f_y = 2y = \lambda g_y = \lambda \cdot 4y^3 \qquad (2)$$
$$f_z = 2z = \lambda g_z = \lambda \cdot 4z^3. \qquad (3)$$

We split this up into 3 possible cases:

Case 1: None of x, y, z are 0.
Case 2: Exactly one of x, y, z, are 0.
Case 3: Exactly two of x, y, z are 0.

Note, not all of x,y,z can be 0 because of the constraint eqn.

Case 1 We can divide each eqn (1), (2), (3) by $4x^3, 4y^3, 4z^3$ resp. to solve for λ. This gives

$$\lambda = \frac{1}{2x^2} = \frac{1}{2y^2} = \frac{1}{2z^2}$$

$\Rightarrow$ $2x^2 = 2y^2 = 2z^2 \Rightarrow x^2 = y^2 = z^2$. Plugging this into the constraint, we get $3x^4 = 1 \Rightarrow x = \pm 3^{-1/4}$. This gives points $(\pm 3^{-1/4}, \pm 3^{-1/4}, \pm 3^{-1/4})$ [all 8 combinations of +/−]. In any case, this gives an f-value of $\sqrt{3}$.

Case 2: The squares of the two nonzero variables must be equal, with common value $1/\sqrt{2}$, and the corresponding value of f is $\sqrt{2}$.

Case 3 The nonzero variable must be ± 1, giving an f-value of 1.

We conclude the max is $\sqrt{3}$ and the min is 1 on $x^4 + y^4 + z^4 = 1$.

Example 5 Find the extreme values of f on the given region.

$$f(x,y) = e^{-xy}, \qquad x^2 + 4y^2 \le 1.$$

Solution For the interior we find critical points of f, and on the boundary we can use Lagrange multipliers

$$f_x = -y e^{-xy}, \quad f_y = -x e^{-xy}.$$

So $f_x = f_y = 0$ if $x = y = 0$. Thus $(0,0)$ is the only CP, and $f(0,0) = 1$.
For the boundary

$$-ye^{-xy} = \lambda \cdot 2x \qquad (1)$$
$$-xe^{-xy} = \lambda \cdot 8y \qquad (2)$$

Note (2) says $y = 0 \Rightarrow x = 0$ and (1) says $x = 0 \Rightarrow y = 0$ So (1) gives

$$-e^{-xy} = \frac{2\lambda x}{y}, \text{ and subbing into (2),}$$

$$\frac{2\lambda x^2}{y} = \lambda 8y \Rightarrow x^2 = 4y^2.$$

Using the constraint, $1 = x^2 + 4y^2 = 2x^2 \Rightarrow x = \pm 1/\sqrt{2}$, and $y = \pm \frac{1}{2\sqrt{2}}$. Now $\underbrace{f(\pm \tfrac{1}{\sqrt{2}}, \mp \tfrac{1}{2\sqrt{2}}) = e^{1/4}}_{\text{maxima}}$ and $\underbrace{f(\pm \tfrac{1}{\sqrt{2}}, \pm \tfrac{1}{2\sqrt{2}}) = e^{-1/4}}_{\text{minima.}}$