15.1 Double integrals over rectangles

In Calc 1, if $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx$$

denotes the area under the graph of f for $a \leq x \leq b$.

Goal: Develop a similar notion for volume under a surface.

Consider $z = f(x,y)$ which is defined on the rectangle

$$R = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

and suppose $f(x,y) \geq 0$ for $(x,y) \in R$. We want to find the volume of the solid S that lies above R and beneath the graph of f. That is,

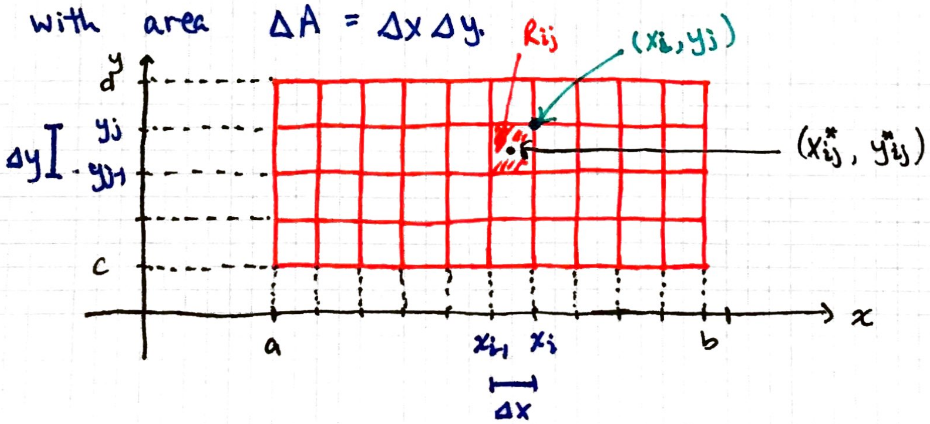
$$S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x,y),\ (x,y) \in R\}.$$

We start by approximating with a Riemann sum, much like calc 1.

First, we divide R into subrectangles: We divide the interval $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ each of width $\Delta x = (b-a)/m$. We divide the interval $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ each of width $\Delta y = (d-c)/n$. Doing this gives us subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x,y) \mid x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}.$$

each with area $\Delta A = \Delta x \Delta y$.



Next, we choose a sample point in each R_{ij}, call it (x_{ij}^*, y_{ij}^*). Then we can approximate the part of S lying above each R_{ij} by a small rectangular box with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$. (Akin to using rectangles) The volume of this box is

$$f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

Thus we can approximate the volume of S by

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

As m, n become large, our approximation gets better. In fact,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

So we define the double integral over R to be

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$

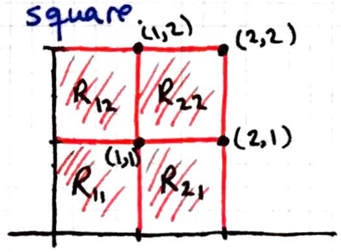
if this limit exists. If the limit does exist, we say that f is integrable.

Remark This definition for the double integral is valid even when f is allowed to be negative. When $f(x,y) \ge 0$, we interpret the double integral as the volume of the solid that lies above R and below the graph of f.

When choosing a sample point (x_{ij}^*, y_{ij}^*), we may as well make it simple and always choose the upper right hand corner (x_i, y_j).

Ex 1 Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Use 4 equal squares and sample upper right corner of each square.

Solution



Here, $\Delta A = 1$ and $m = n = 2$

So,

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\,\Delta A$$

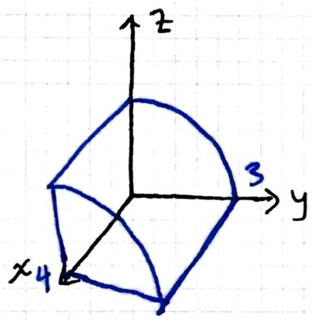
$$= f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1) = 34.$$

Note Could also use this method with midpoints as sample points. We denote $\bar{x}_i$ as the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ as the midpoint of $[y_{j-1}, y_j]$.

Example 2 Evaluate the integral $\iint_R \sqrt{9-y^2}\, dA$ where $R = [0,4] \times [0,3]$. Sketch the solid represented.

Solution We begin by sketching the solid, then use geometry to compute the volume.



This is $\frac{1}{4}$ of a cylinder of radius 3 and height 4. So

$$V = \frac{1}{4} \cdot \pi (3)^2 \cdot 4 = 9\pi.$$

Iterated Integrals

Obviously we need a better way to compute integrals. Suppose f is integrable on $R = [a,b] \times [c,d]$. Then

$$\int_c^d f(x,y)\, dy$$

denotes the partial integral with respect to y. That is, we hold x constant and use the Fundamental Theorem of Calculus to compute the integral. Since each of these depends on x, we actually have a function of x:

$$A(x) = \int_c^d f(x,y)\, dy.$$

Now we have a Calc 1 integral;

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx.$$

The RHS is called an iterated integral. We usually don't write the brackets, so

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx.$$

Similarly,

$$\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy.$$

Example 3 Evaluate the iterated integrals.

(a) $\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx$ (b) $\int_0^2 \int_1^4 (6x^2y - 2x)\, dx\, dy$

Solution

(a) $$\int_1^4 \left[\int_0^2 (6x^2y - 2x)\, dy \right] dx = \int_1^4 \left[3x^2y^2 - 2xy \right]_{y=0}^{y=2} dx$$

$$= \int_1^4 (12x^2 - 4x)\, dx$$

$$= 4x^3 - 2x^2 \Big]_1^4 = 222$$

(b) $$\int_0^2 \left[\int_1^4 (6x^2y - 2x)\, dx \right] dy = \int_0^2 \left[2x^3y - x^2 \right]_{x=1}^{x=4} dy$$

$$= \int_0^2 (126y - 15)\, dy$$

$$= 63y^2 - 15y \Big]_0^2 = 222.$$

Remark It's good practice to label the bounds of the inside integral to remind yourself which variable you plug them in to.

It's not a coincidence that these integrals are the same:

Fubini's Theorem If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

More generally, we just need f to be integrable.

Example 4 Evaluate $I = \iint_R \frac{x}{1+xy}\, dA$, where $R = [0,1]\times[0,1]$.

Solution By Fubini, it doesn't matter if we do "dxdy" or "dydx".

$$I = \int_0^1\int_0^1 \frac{x}{1+xy}\, dy\, dx = \int_0^1 \int_{y=0}^{y=1} \frac{1}{u}\, du\, dx \qquad \begin{array}{l} u = 1+xy \\ du = x\, dy \end{array}$$

$$= \int_0^1 \Big[\log(1+xy)\Big]_{y=0}^{y=1} dx$$

$$= \int_0^1 \log(1+x)\, dx = (1+x)\log(1+x) - x\Big]_0^1 \qquad \text{(integration by parts)}$$

$$= 2\log 2 - 1.$$

Suppose $f(x,y) = g(x)h(y)$. Then Fubini's Theorem gives

$$\iint_R f(x,y)\, dA = \int_a^b\int_c^d f(x,y)\, dy\, dx$$

$$= \int_c^d\left[\int_a^b g(x)\, h(y)\, dx\right]dy$$

$$= \int_c^d\left[h(y)\int_a^b g(x)\, dx\right]dy \qquad h(y) \text{ constant wrt } x.$$

$$= \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right) \qquad \int_a^b g(x)\, dx \text{ is a constant.}$$

Thus if $f(x,y) = g(x)h(y)$ we can factor the integral.

Example 5

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin x \cos y\, dx\, dy = \int_0^{\pi/2} \sin x\, dx \int_0^{\pi/2} \cos y\, dy$$

$$= 1.$$