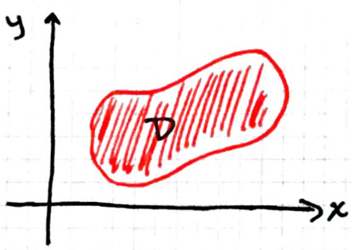
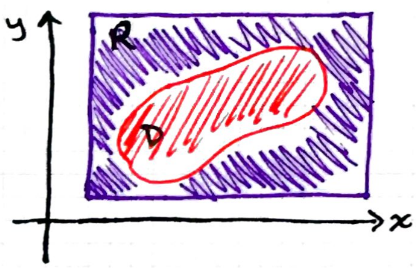
15.2 Double integrals over general regions

For a single-variable function, we always integrate over an interval. But rectangles aren't the only types of regions beneath a surface.

Suppose D is a bounded (and nice enough) region, and $f(x,y)$ is defined and continuous on D. We define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D, \end{cases}$$





where R is a rectangular region enclosing D. If F is integrable over R, then we define the double integral of f over D by

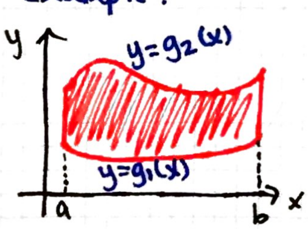
$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA.$$

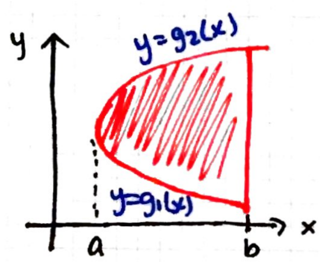
Since $F(x,y) = 0$ in the part of R outside of D, this is a reasonable definition. We want to focus on two types of regions.

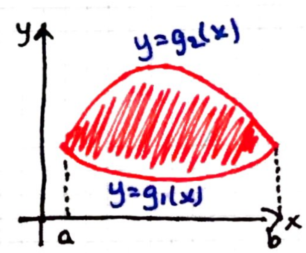
D is of type I if it lies between the graphs of two continuous functions of x, i.e.,

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

For example:







To evaluate such an integral, we choose a rectangle $R = [a,b] \times [c,d]$ containing D. Then by Fubini,

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

But since $F(x,y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$,

$$\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

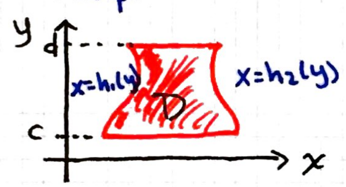
Since $F(x,y) = f(x,y)$ for $g_1(x) \le y \le g_2(x)$.

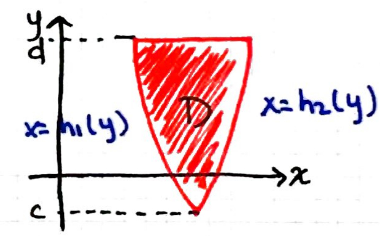
Thus to integrate over D of type I, we have

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

D is of type II if $D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

For example:





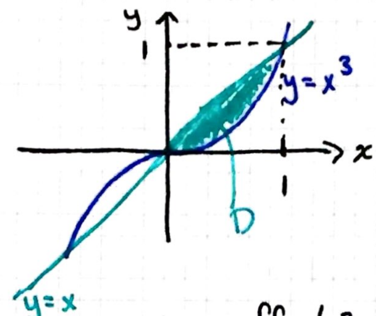
By the same reasoning we have in this case

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

<u>Remark</u> You need to draw the region of integration to give yourself (and whoever reads your work) a clear picture of what is happening.

<u>Example 1</u> Evaluate $\iint_D (x^2+2y)\,dA$, where D is bounded by $y=x$, $y=x^3$, $x \ge 0$.

Solution We first draw D.



(We only want the space between them with $x \geq 0$.)

Note that the two curves intersect at (0,0) and (1,1). Moreover, D is of type I, so,

$$\iint_D (x^2+2xy)\,dA = \int_0^1 \int_{x^3}^{x} (x^2+2xy)\,dy\,dx$$

$$= \int_0^1 \left[x^2y + xy^2\right]_{y=x^3}^{y=x} dx$$

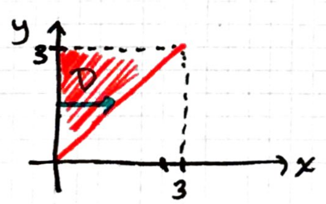
$$= \int_0^1 \left[(x^3+x^3)-(x^5+x^7)\right] dx$$

$$= \int_0^1 (2x^3 - x^5 - x^7)\,dx = \frac{1}{2}x^4 - \frac{1}{6}x^6 - \frac{1}{8}x^8 \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{6} - \frac{1}{8} = \frac{5}{24}.$$

Example 2 $\iint_D e^{-y^2}\,dA$, where $D = \{(x,y) \mid 0 \leq y \leq 3,\ 0 \leq x \leq y\}$

Solution



Since x ranges from 0 to y,

$$\int_0^3 \int_0^y e^{-y^2}\,dx\,dy = \int_0^3 y e^{-y^2}\,dy$$

$$= -\frac{1}{2}e^{-y^2}\Big|_0^3$$

$$= -\frac{1}{2}e^{-9} + \frac{1}{2}.$$

Remark e^{-y^2} doesn't have an antiderivative in terms of elementary functions, yet we can integrate it in some contexts in two variables!

Some regions D are both type I and type II. In this case, the order of integration we choose depends on what is easier (or possible). The region D is also type I in Example 2, so we could have written

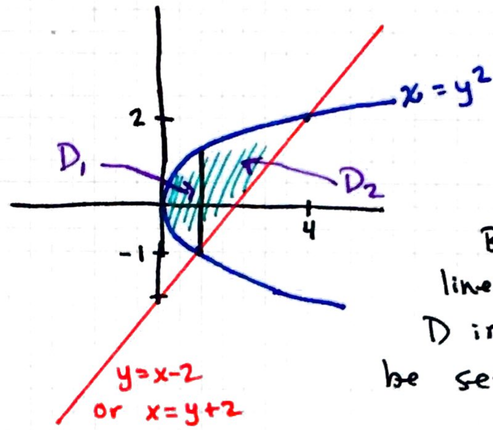
$$\int_0^3 \int_x^3 e^{-y^2}\,dy\,dx.$$

But as we mentioned, e^{-y^2} has no antiderivative, so this integral is impossible.

Example 3 Set up both orders of integration

$\iint_D y\, dA$, D bounded by $y = x-2$, $x = y^2$

Solution



Notice this is a type II region. So we can write

$$\int_{-1}^{2} \int_{y^2}^{y+2} y\, dx\, dy.$$

But if we draw a vertical line at $x=1$, we have divided D into D_1 and D_2, which can both be seen as type I regions.

Now $x = y^2$ separates into $y = \sqrt{x}$ and $y = -\sqrt{x}$. So

$$\iint_{D_1} y\, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\, dy\, dx \quad \text{and}$$

$$\iint_{D_2} y\, dA = \int_1^4 \int_{x-2}^{\sqrt{x}} y\, dy dx$$

As you might expect, $\iint_D y\, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\, dy dx + \int_1^4 \int_{x-2}^{\sqrt{x}} y\, dy dx.$

In this case we would rather do the first integral since it is much less work.

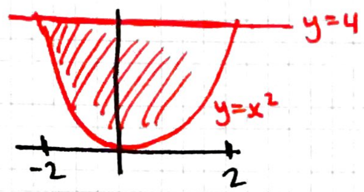
Properties of double integrals

1) $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$
2) $\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA$, where $c \in \mathbb{R}$.
3) If $f(x,y) \geq g(x,y)$ in D, then $\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$
4) $\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$, where $D = D_1 \cup D_2$ and D_1, D_2 do not overlap except for their boundaries
5) $\iint_D 1\, dA = A(D)$, where $A(D) =$ Area of D.
6) If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$, then

$$m A(D) \leq \iint_D f(x,y)\, dA \leq M \cdot A(D).$$

Example 4 Find the volume enclosed by the cylinders $z=x^2$, $y=x^2$, and the planes $z=0$, $y=4$.

Solution



$$V = \int_{-2}^{2}\int_{x^2}^{4} x^2 \, dy\, dx$$
$$= \int_{-2}^{2}(4x^2 - x^4)\, dx$$
$$= 2\int_0^2 (4x^2 - x^4)\, dx$$
$$= 2\left(\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right)\Big|_0^2$$
$$= 2\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \tfrac{128}{15}.$$

Average Value

Recall the average value of $y=f(x)$ on $[a,b]$ is

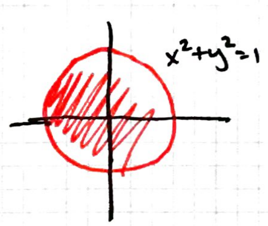
$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

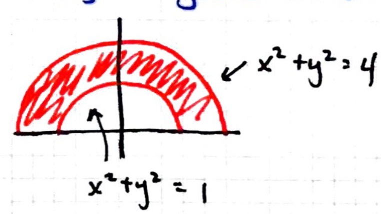
Analogously, if we want the average value of $z=f(x,y)$ on D, we compute

$$f_{ave} = \frac{1}{A(D)}\iint_D f(x,y)\, dA.$$

15.3 Double integrals in polar coordinates.

Given a region with rotational symmetry, sometimes its convenient to use polar coordinates. Eg. regions like





$R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ $\qquad$ $R = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

Here we are using $x^2+y^2=r^2$, $x = r\cos\theta$, $y = r\sin\theta$.

The region $R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$ is called a polar rectangle.

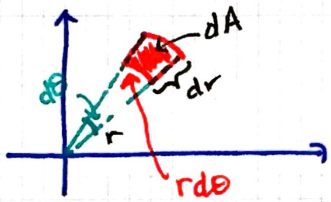
Changing to polar coordinates in a double integral:

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

Notice There is a factor of r in the polar integral!

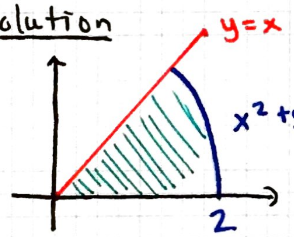
Where does the r come from?



So $dA = (r\,d\theta)(dr) = r\,dr\,d\theta$.

Example 5 Evaluate $\iint_R (2x-y)\,dA$ where R is the region enclosed by the circle $x^2+y^2=4$, the lines $y=0$, $y=x$, in the first quadrant.

Solution



$R = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \pi/4\}$

$$\iint_R (2x-y)\,dA = \int_0^{\pi/4}\int_0^2 (2r\cos\theta - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^2 (2r^2\cos\theta - r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/4} (2\cos\theta - \sin\theta)\,d\theta \int_0^2 r^2\,dr$$

$$= 2\sin\theta + \cos\theta\Big|_0^{\pi/4} + \tfrac{1}{3}r^3\Big|_0^2$$

$$= (3/\sqrt{2} - 1)(\tfrac{8}{3})$$

$$= 4\sqrt{2} - \tfrac{8}{3}.$$

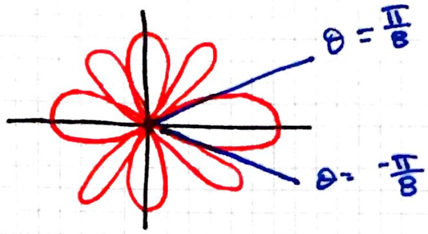
If f is continuous on a polar region of the form

$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example 6 Use a double integral to find area enclosed by one loop of $r = \cos 4\theta$.

Solution



$$D = \{(x,y) \mid -\tfrac{\pi}{8} \le \theta \le \tfrac{\pi}{8},\ 0 \le r \le \cos 4\theta\}$$

So area is

$$A(D) = \iint_D dA = \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \int_0^{\cos 4\theta} r\, dr\, d\theta$$

$$= \int_{-\pi/8}^{\pi/8} \left[\frac{1}{2} r^2\right]_{r=0}^{r=\cos 4\theta} d\theta$$

$$= \int_{-\pi/8}^{\pi/8} \frac{1}{2} \cos^2 4\theta\, d\theta$$

$$= \frac{1}{4} \int_{-\pi/8}^{\pi/8} (1 + \cos 8\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{8} \sin 8\theta\right]_{-\pi/8}^{\pi/8} = \frac{\pi}{16}$$