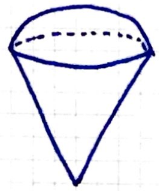
15.3 Double integrals in polar coords (cont'd)

Ex 1 Use polar coordinates to find the volume of the solid above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.

Solution The described region is something like an ice cream cone



For a precise picture, look under "For your reference" on the webpage.

The cone intersects the sphere at

$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=1 \iff x^2+y^2=\tfrac{1}{2}.$$

So

$$V=\iint_{x^2+y^2\le\frac12}\left(\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\right)dA$$

(The volume under the sphere minus that under the cone)

$$=\int_0^{2\pi}\int_0^{1/\sqrt2}\left(\sqrt{1-r^2}-r\right)r\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{1/\sqrt2}\left(r\sqrt{1-r^2}-r^2\right)dr=\frac{\pi}{3}\left(2-\sqrt2\right)$$

Example 2 Show that $\int_{-\infty}^{\infty}e^{-x^2}\,dx=\sqrt\pi$.

Solution The trick is to use polar coordinates. If D_a is the disk of radius a centered at the origin, then

$$\iint_{D_a}e^{-(x^2+y^2)}\,dA=\int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta=\pi\left(1-e^{-a^2}\right).$$

Now $\lim_{a\to\infty}\pi\left(1-e^{-a^2}\right)=\pi$. On the other hand,

$$\lim_{a\to\infty}\iint_{D_a}e^{-(x^2+y^2)}\,dA=\iint_{\mathbb{R}^2}e^{-(x^2+y^2)}\,dA=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)}\,dx\,dy.$$

But

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)}\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-x^2}e^{-y^2}\,dx\,dy$$

$$=\int_{-\infty}^{\infty}e^{-x^2}\,dx\int_{-\infty}^{\infty}e^{-y^2}\,dy$$

$$=\int_{-\infty}^{\infty}e^{-x^2}\,dx\int_{-\infty}^{\infty}e^{-x^2}\,dx$$ [Since variable of integration is arbitrary]

$$=\left(\int_{-\infty}^{\infty}e^{-x^2}\,dx\right)^2.$$

Thus, square-rooting both sides, $\int_{-\infty}^{\infty}e^{-x^2}\,dx=\sqrt\pi$.

15.5 Applications of double integrals.

In this section we will only do center of mass and first moment. In the book, there are applications to probability which are interesting in their own right.

Density and Mass

Recall A *lamina* is a thin plate. We will consider laminas with variable density, that is, $\rho(x,y)$ is a continuous function on a lamina D.

Since density is mass/volume, if we subdivide D into several rectangles R_{ij} of the same size ΔA, and pick a point (x_{ij}^*, y_{ij}^*) in each R_{ij}, then the mass of D is

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A,$$

Now letting the number of rectangles $\to \infty$, we see

$$\boxed{m = \iint_D \rho(x,y)\, dA.}$$

Recall The *moment* of a particle about an axis is the product of its mass and its directed distance from the axis. Reasoning similar to that of above shows that the moment of the lamina D about the x-axis is

$$\boxed{M_x = \iint_D y\, \rho(x,y)\, dA.}$$

Similarly, the moment about the y-axis is

$$\boxed{M_y = \iint_D x\, \rho(x,y)\, dA.}$$

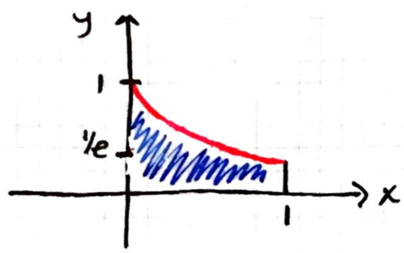
We define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$.

Note that

$$\boxed{\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\, dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint y\,\rho(x,y)\, dA.}$$

Example 3 Find the mass and center of mass of the lamina that occupies the region D = the region bounded by $y=e^{-x}$, $y=0$, $x=0$, $x=1$ and has the density function $\rho(x,y) = xy$.

Solution We graph the region D:



$$m = \iint_D \rho(x,y)\, dA = \int_0^1 \int_0^{e^{-x}} xy \, dy\, dx$$

$$= \int_0^1 \left[\tfrac{1}{2} xy^2\right]_{y=0}^{y=e^{-x}} dx$$

$$= \tfrac{1}{2}\int_0^1 xe^{-2x}\, dx \qquad u=x \quad dv = e^{-2x} \quad du = dx \quad v = -\tfrac{1}{2}e^{-2x}$$

$$= \tfrac{1}{2}\left[-\tfrac{1}{2}xe^{-2x} + \tfrac{1}{2}\int e^{-2x}\,dx\right]_0^1$$

$$= \tfrac{1}{2}\left[-\tfrac{1}{2}xe^{-2x} - \tfrac{1}{4}e^{-2x}\right]_0^1 = \boxed{\tfrac{1}{8} - \tfrac{3}{8}e^{-2}}$$

$$M_y = \int_0^1\int_0^{e^{-x}} x^2 y\, dy\, dx$$

$$= \int_0^1 \left[\tfrac{1}{2}x^2y^2\right]_{y=0}^{y=e^{-x}} dx = \tfrac{1}{2}\int_0^1 x^2 e^{-2x}\, dx \quad \text{[by parts twice]}$$

$$= \boxed{\tfrac{1}{8} - \tfrac{5}{8}e^{-2}}$$

$$M_x = \int_0^1\int_0^{e^{-x}} xy^2\, dy\, dx = \int_0^1 \left[\tfrac{1}{3}xy^3\right]_{y=0}^{y=e^{-x}} dx$$

$$= \tfrac{1}{3}\int_0^1 xe^{-3x}\, dx \quad \text{[by parts]}$$

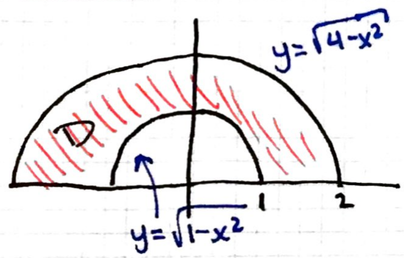
$$= \boxed{\tfrac{1}{27} - \tfrac{4}{27}e^{-3}}$$

So $m = \frac{1}{8}(1-3e^{-2})$, $(\bar{x},\bar{y}) = \left(\frac{(1/8)(1-5e^{-2})}{(1/8)(1-3e^{-2})}, \frac{(1/27)(1-4e^{-3})}{(1/8)(1-3e^{-2})}\right)$

$$= \left(\frac{e^2-5}{e^2-3}, \frac{8(e^3-4)}{27(e^3-3e)}\right)$$

Example 4 Find the mass of the lamina whose boundary consists of the semicircles $y=\sqrt{1-x^2}$ and $y=\sqrt{4-x^2}$ together with the portions of the x-axis that joins them and whose density is inversely proportional to its distance from the origin.

Solution

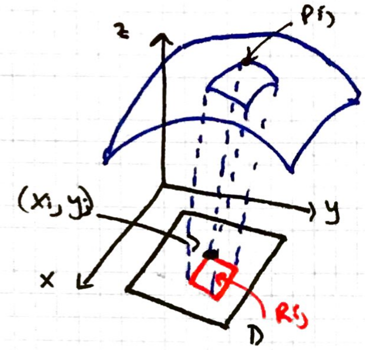


$$\rho(x,y) = \frac{k}{\sqrt{x^2+y^2}} = \frac{k}{r}$$

$$m = \iint_D \rho(x,y)\, dA = \int_0^{\pi}\int_1^2 \left(\frac{k}{r}\right) r\, dr\, d\theta$$

$$= k\int_0^{\pi} d\theta \int_1^2 dr = \boxed{\pi k}$$

15.5 Surface area



We want to compute the area of a surface $z = f(x,y)$ above a rectangle D. We'll assume $f(x,y) \geq 0$ on D. We'll approximate the area ΔS_{ij} of a patch lying above the rectangle R_{ij} by the area ΔT_{ij} of the tangent plane lying directly above S_{ij}. If $\vec{a}, \vec{b}$ are the vectors starting at P_{ij} along the sides of the parallelogram T_{ij}, then $\Delta T_{ij} = |\vec{a} \times \vec{b}|$.

Now

$$\vec{a} = \langle \Delta x, 0, f_x(x_i, y_j) \rangle, \quad \vec{b} = \langle 0, \Delta y, f_y(x_i, y_j)\Delta y \rangle$$

So
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & f_x(x_i, y_j)\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\Delta y \end{vmatrix}$$

$$= \langle -f_x(x_i, y_j)\Delta x \Delta y, -f_y(x_i, y_j)\Delta x \Delta y, \Delta x \Delta y \rangle$$
$$= \langle -f_x(x_i, y_j), -f_y(x_i, y_j), 1 \rangle \Delta A$$

Thus $\Delta T_{ij} = |\vec{a} \times \vec{b}| = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\ \Delta A$

Taking limits in the double sum over i, j.

$$\boxed{A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\ dA.}$$

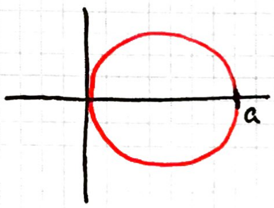
Example 5 Find the area of the surface which is the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies above the cylinder $x^2 + y^2 = ax$ and lies above the xy-plane.

Solution

$f(x,y) = \sqrt{a^2 - x^2 - y^2} \implies f_x = -x(a^2-x^2-y^2)^{-1/2}, \quad f_y = -y(a^2-x^2-y^2)^{-1/2}$

We use polar coordinates.

$x^2 + y^2 = ax \implies r = a\cos\theta$, so



$D = \{(r,\theta) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le a\cos\theta\}$

$$A(S) = \iint_D \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2} + 1}\, dA$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \sqrt{\frac{r^2}{a^2-r^2} + 1}\; r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{a\cos\theta} \frac{ar}{\sqrt{a^2-r^2}}\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[-a\sqrt{a^2-r^2}\right]_{r=0}^{r=a\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2-a^2\cos^2\theta} - a\right) d\theta = 2a^2 \int_0^{\pi/2} \left(1 - \sqrt{1-\cos^2\theta}\right) d\theta$$

$$= 2a^2 \int_0^{\pi/2} \left(1 - \sqrt{\sin^2\theta}\right) d\theta = 2a^2 \int_0^{\pi/2} (1 - \sin\theta)\, d\theta$$

$$= a^2(\pi - 2).$$