15.6 Triple integrals

Just as we defined the double integral for functions of two variables, we can define triple integrals for functions of three variables. Instead of being defined over a rectangle, $w = f(x,y,z)$ is defined over a rectangular box, B. We form an analogous Riemann sum.

Fubini's Theorem If f is continuous on $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

Note: For functions of two variables we had two possible orders of integration, namely, $dx\,dy$ or $dy\,dx$. With three variables there are $3! = 6$ possible orders.

In 3 dimensions, we typically label a general bounded region by E, and D its projection onto one of the planes. For example, if D lies in the xy plane and

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\},$$

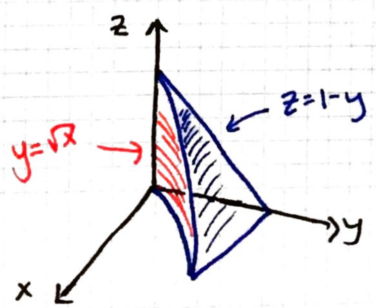
then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA.$$

Example 1 Express the following integral in the other 5 orders.

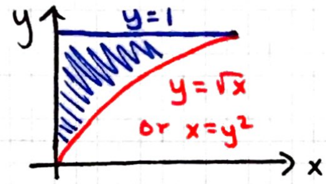
$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx.$$

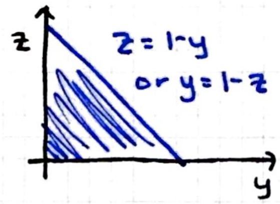
Solution $E = \{(x,y,z) \mid 0 \le z \le 1-y,\ \sqrt{x} \le y \le 1,\ 0 \le x \le 1\}$

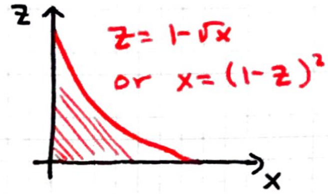


We project E onto the xy-, yz- and xz-planes. For the xy-plane, $z=0$, which gives the boundary curves $y=1$ and $y=\sqrt{x}$. For the yz-plane, $x=0$, which gives boundary curves $z=0$, $z=1-y$, $y=0$. And for the xz-plane, since $\sqrt{x} \le y$, $0 \le z \le 1-\sqrt{x}$ and $0 \le x \le 1$.

Drawing these regions:







Now for the inner integrals, if we consider x first, x ranges from the surface $x=0$ to $x=y^2$, and y ranges from $y=\sqrt{x}$ to $y=1-z$ if we consider y first. Putting this together.

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$

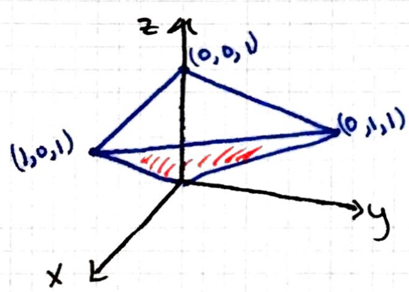
$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

Example 2 Evaluate $\iiint_T xz\,dV$, where T is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,1)$, $(0,1,1)$ and $(0,0,1)$.

Solution

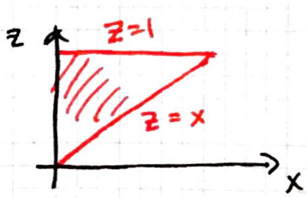


T is bounded by $x=0$, $y=0$ and $z=1$. We need to find the equation of the plane for the other bound. (in red) Note that the vectors $\vec{a}=\langle 1,0,1\rangle$ and $\vec{b}=\langle 0,1,1\rangle$ span the plane, so

$\vec{n}=\langle 1,0,1\rangle\times\langle 0,1,1\rangle=\langle -1,-1,1\rangle$ is

a normal vector. So an equation is $-x-y+z=0$ or $z=x+y$. Thus $T=\{(x,y,z)\mid 0\le y\le z-x,\ \underbrace{x\le z\le 1,\ 0\le x\le 1}\}$

Projecting onto xz-plane



So

$$\iiint_T xz\,dV = \int_0^1\int_x^1\int_0^{z-x} xz\,dy\,dz\,dx$$

$$= \int_0^1\int_x^1 xz(z-x)\,dz\,dx = \int_0^1\int_x^1 (xz^2 - x^2z)\,dz\,dx$$

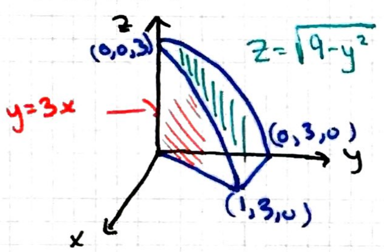
$$= \int_0^1 \left[\tfrac{1}{3}xz^3 - \tfrac{1}{2}x^2z^2\right]_{z=x}^{z=1} dx$$

$$= \int_0^1 \left(\tfrac{1}{3}x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^4 + \tfrac{1}{2}x^4\right) dx$$

$$= \tfrac{1}{6}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{30}x^5 \Big|_0^1 = \tfrac{1}{30}.$$

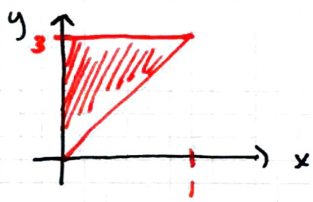
Example 3 Set up an integral $\iiint_E f(x,y,z)\,dV$, where E is bounded by the cylinder $y^2+z^2=9$, planes $x=0$, $y=3x$, $z=0$ in first octant.

Solution



z ranges from $z=0$ to $z=\sqrt{9-y^2}$.

Projecting onto the xy-plane,



So y ranges from $y=3x$ to $y=3$, x ranges from 0 to 1. So

$$\iiint_E f(x,y,z)\,dV = \int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dy\,dx.$$

Just as $\iint_D 1\,dA = A(D)$, we have that $\iiint_E 1\,dV = \mathrm{Vol}(E)$.

Example 4 Use a triple integral to find the volume of the solid enclosed by the paraboloids $y=x^2+z^2$ and $y=8-x^2-z^2$.

Solution The two paraboloids intersect when $x^2+z^2 = 8-x^2-z^2$ $\Leftrightarrow$ $x^2+z^2=4$, the circle of radius 2, center $(0,4,0)$.

Thus the projection of E onto the xz-plane is $D=\{(x,z) \mid x^2+z^2 \le 4\}$.

And $E = \{(x,y,z) \mid x^2+z^2 \le y \le 8-x^2-y^2,\ x^2+z^2 \le 4\}$.

So

$$V = \iiint_E dV = \iint_D \left(\int_{x^2+z^2}^{8-x^2-z^2} dy \right) dA$$

$$= \iint_D 8 - 2(x^2+z^2)\, dA.$$

$$= \int_0^{2\pi} \int_0^2 (8 - 2r^2)\, r\,dr\,d\theta$$

Polar coords:
$x = r\cos\theta$
$z = r\sin\theta$

$$= \int_0^{2\pi} d\theta \int_0^2 (8r - 2r^3)\,dr$$

$$= 16\pi.$$