15.7 Cylindrical Coordinates

In the cylindrical coordinate system, a Point P is represented by an ordered triple (r, θ, z), where (r, θ) represents the projection of P onto the xy-plane in polar coordinates.

Thus to convert from cylindrical coordinates to rectangular coordinates we use the equations:

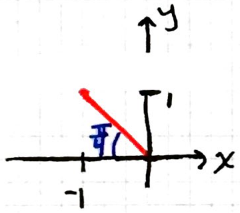
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

and to convert from rectangular coordinates to cylindrical coordinates, we use the equations:

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x, \quad z = z.$$

Example 1 Write the point $(-1,1,2)$ in cylindrical coordinates.

Solution We have $z = 2$. Projecting onto xy-plane,



$r^2 = x^2 + y^2 = (-1)^2 + (1)^2 = 2 \Rightarrow r = \sqrt{2}$,

$\tan^{-1}(1) = \pi/4$, so $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$,

So $(r, \theta, z) = (\sqrt{2}, \frac{3\pi}{4}, 2)$.

Example 2 Write the following equation in cylindrical coordinates.

$$3x^2 + 3y^2 + x + y - z^2 = 8$$

Solution $3(x^2+y^2) + x + y - z^2 = 3r^2 + r\cos\theta + r\sin\theta - z^2 = 8.$

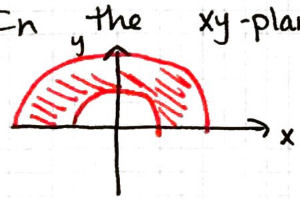
Example 3 Sketch the solid described by

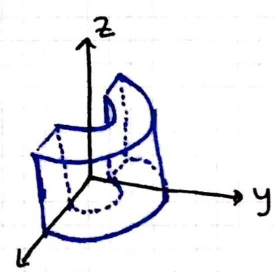
(a) $1 \le r \le 2, \quad 0 \le \theta \le \pi, \quad 0 \le z \le 1$

(b) $0 \le r \le 1, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le r^2$

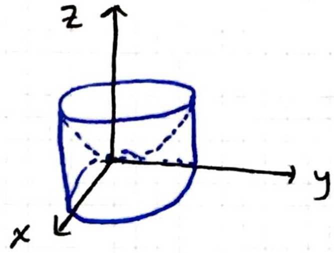
Solution

(a) In the xy-plane:





(b) In the xy plane, we have a disk of radius 1. Now $z = r^2 = x^2+y^2$ is a paraboloid, so



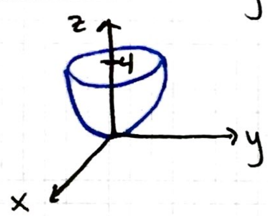
(cylinder with the paraboloid removed)

Integrals in cylindrical coordinates

Since we are essentially converting to polar coords in the xy-plane, we have $dV = r\,dr\,d\theta\,dz = r\,dz\,dr\,d\theta$

Example 4 Evaluate $\iiint_E z\,dV$, where E is enclosed by the paraboloid $z = x^2+y^2$ and the plane $z=4$.

Solution Drawing E:



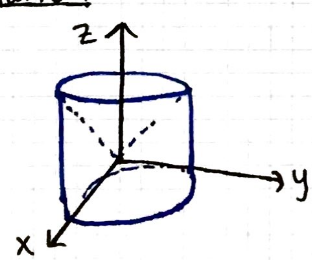
$r^2 = x^2+y^2 \le z \le 4$, and $0 \le r \le 2$, $0 \le \theta \le 2\pi$,

so

$$\iiint_E z\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 zr\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 \left[\tfrac{1}{2}z^2 r\right]_{z=r^2}^{z=4} dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr$$
$$= 2\pi\left(16 - 16/3\right)$$
$$= \tfrac{64}{3}\pi.$$

Example 5 Evaluate $\iiint_E x^2\,dV$, where E is the solid that lies within the cylinder $x^2+y^2=1$ above the plane $z=0$ and below the cone $z = 4x^2+4y^2$

Solution



Here z is bounded above by $2\sqrt{x^2+y^2} = 2r$

$$\iiint_E x^2\,dV = \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\, r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\,dr\,d\theta$$
$$= \int_0^{2\pi}\cos^2\theta\,d\theta \int_0^1 2r^4\,dr$$
$$= \int_0^{2\pi}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta \int_0^1 2r^4\,dr$$
$$= \left[\tfrac{1}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}\right]\left[\tfrac{2}{5}r^5\Big|_0^1\right] = \frac{2\pi}{5}.$$

<u>Example 6</u> Evaluate the integral by changing to cylindrical coordinates.

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx$$

<u>Solution</u> Here $0 \le z \le 9-(x^2+y^2) \Rightarrow 0 \le z \le 9-r^2$

$0 \le y \le \sqrt{9-x^2} \Rightarrow 0 \le x^2+y^2 \le 9 \Rightarrow r^2 \le 9$

$-3 \le x \le 3$

So since $y \ge 0$, in the xy plane we have the upper half-circle of radius 3, and z lies between $z=0$ and the paraboloid.

The integral becomes

$$\int_0^{\pi} \int_0^{3} \int_0^{9-r^2} r^2\, dz\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{3} (9-r^2) r^2\, dr\, d\theta = \int_0^{\pi} d\theta \int_0^{3} (9r^2 - r^4)\, dr$$

$$= \pi \left[3r^3 - \tfrac{1}{5} r^5\right]_0^3 = \pi\left[81 - \tfrac{243}{5}\right]$$

$$= \frac{162}{5}\pi.$$