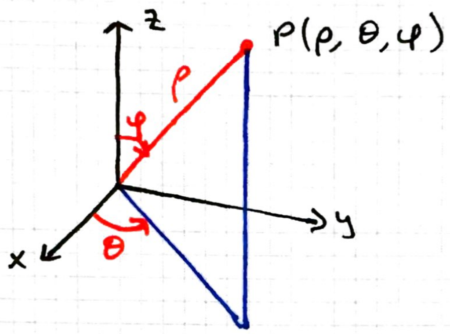
15.8 Triple integrals in spherical coordinates

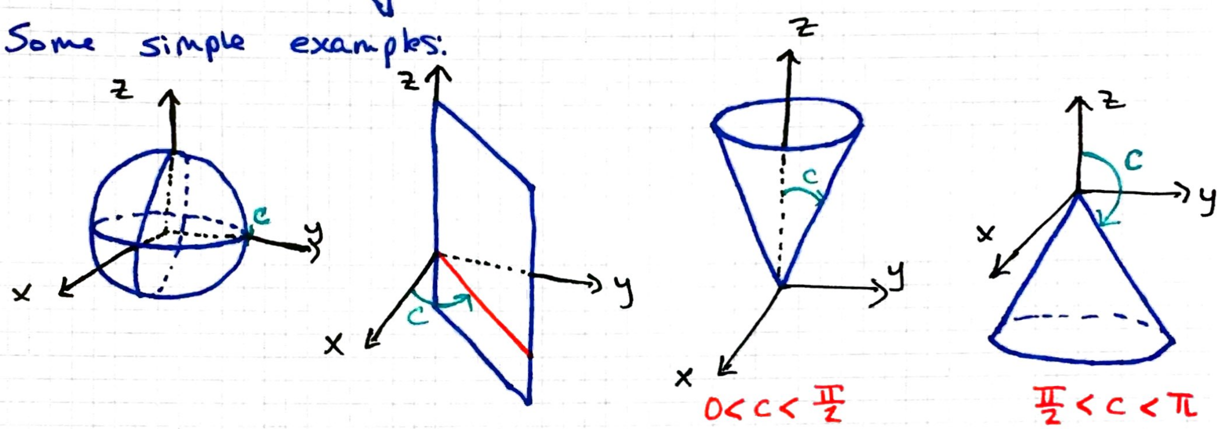
We can represent $P(x,y,z)$ in spherical coords (ρ,θ,φ), where $\rho = |OP|$ (distance from origin), θ is same angle from cylindrical coords, and φ is the angle between the positive z-axis and the segment OP.

Note: $\rho \geq 0$, $0 \leq \varphi \leq \pi$



These coords are useful when there is symmetry about a point.

Some simple examples:



$\rho = c$ sphere

$\theta = c$ half-plane

$\varphi = c$ half-cone

A little bit of high school trigonometry shows

$$\boxed{\begin{gathered} x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi \\ x^2+y^2+z^2 = \rho^2 \end{gathered}}$$

Note also that $\tan\theta = y/x$, $\rho = \sqrt{x^2+y^2+z^2}$, $\cos\varphi = z/\rho$ allows us to obtain spherical coords from rectangular coords.

Example 1 Find rectangular coordinates of the point given in spherical coordinates.

(a) $(2, \pi/2, \pi/2)$ (b) $(4, -\frac{\pi}{4}, \frac{\pi}{3})$

Solution

(a) $x = 2\sin\frac{\pi}{2}\cos\frac{\pi}{2} = 0$, $y = 2\sin\frac{\pi}{2}\sin\frac{\pi}{2} = 2$, $z = 2\cos\frac{\pi}{2} = 0$

$(0, 2, 0)$

(b) $x = 4\sin\frac{\pi}{3}\cos\frac{-\pi}{4} = 4\,\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = 2\sqrt{3/2} = \sqrt{6}$

$y = 4\sin\frac{\pi}{3}\sin\frac{-\pi}{4} = 4\cdot\frac{\sqrt{3}}{2}\left(\frac{-1}{\sqrt{2}}\right) = -2\sqrt{3/2} = -\sqrt{6}$

$z = 4\cos\frac{\pi}{3} = 4\cdot\frac{1}{2} = 2$

$(\sqrt{6}, -\sqrt{6}, 2)$

Example 2 Change from rectangular to spherical coords.

(a) $(1, 0, \sqrt{3})$ (b) $(\sqrt{3}, -1, 2\sqrt{3})$

Solution

(a) $\rho = \sqrt{1^2 + 0 + \sqrt{3}^2} = 2$, $\tan\theta = 0/1 \Rightarrow \theta = 0$

$\cos\varphi = \frac{\sqrt{3}}{2} \Rightarrow \varphi = \pi/6$, so $(\rho, \theta, \varphi) = (2, 0, \pi/6)$

(b) $\rho = \sqrt{3 + 1 + 12} = 4$, $\tan\theta = \frac{-1}{\sqrt{3}} \Rightarrow \theta = \frac{11\pi}{6}$

$\cos\varphi = \frac{\sqrt{3}}{2} \Rightarrow \varphi = \pi/6$, so $(\rho, \theta, \varphi) = (4, \frac{11\pi}{6}, \frac{\pi}{6})$.

Triple integrals

If $E = \{(\rho, \theta, \varphi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \varphi \le d\}$, then

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi}$$

Notice $dV = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$. This can be obtained by methods similar to

Ex3 Write the eqn $z = x^2 + y^2$ in spherical coords.

Solution

$\rho\cos\varphi = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta = \rho^2\sin^2\varphi$

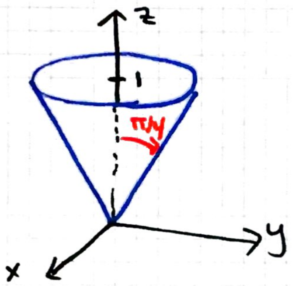
$\Rightarrow \boxed{\rho = \cot\varphi\csc\varphi}$

__Example 4__ Sketch the solid whose volume is given by the following integral and evaluate.

$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\varphi} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

__Solution__ $E = \{(\rho,\theta,\varphi) \mid 0 \le \rho \le \sec\varphi, \; 0 \le \theta \le 2\pi, \; 0 \le \varphi \le \pi/4\}$

$\rho = \sec\varphi \iff \rho\cos\varphi = 1 \iff z = 1$ and $\varphi = \pi/4$ is a cone, so



$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\varphi} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{\pi/4}\int_0^{2\pi} \left[\tfrac{1}{3}\rho^3 \sin\varphi\right]_{\rho=0}^{\rho=\sec\varphi} d\theta \, d\varphi$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi/4} \sec^3\varphi \sin\varphi \, d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi/4} \sec^2\varphi \tan\varphi \, d\varphi$$

$$= \frac{2\pi}{3}\cdot\frac{1}{2}\tan^2\varphi \Big|_0^{\pi/4} = \boxed{\pi/3}$$

__Example 5__ Use spherical coords to evaluate $\iiint_E y^2 \, dV$ where E is the solid hemisphere $x^2+y^2+z^2 \le 9$, $y \ge 0$.

__Solution__ $E = \{(\rho,\theta,\varphi) \mid 0 \le \rho \le 3, \; 0 \le \theta \le \pi, \; 0 \le \varphi \le \pi\}$, so

$$\iiint_E y^2 \, dV = \int_0^{\pi}\int_0^{\pi}\int_0^{3} \rho^2 \sin^2\varphi \sin^2\theta \; \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{\pi} \sin^3\varphi \, d\varphi \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{3} \rho^4 \, d\rho$$

$$= \int_0^{\pi} (1-\cos^2\varphi)\sin\varphi \, d\varphi \int_0^{\pi} \tfrac{1}{2}(1-\cos 2\theta)\, d\theta \int_0^{3} \rho^4 \, d\rho$$

$$= \left[-\cos\varphi + \tfrac{1}{3}\cos^3\varphi\right]_0^{\pi} \left[\tfrac{1}{2}\left(\theta - \tfrac{1}{2}\sin 2\theta\right)\right]_0^{\pi} \left[\tfrac{1}{5}\rho^5\right]_0^{3}$$

$$= \left(\tfrac{2}{3}+\tfrac{2}{3}\right)\left(\tfrac{1}{2}\pi\right)\left(\tfrac{1}{5}(243)\right) = \frac{162\pi}{5}$$

Example 6 Evaluate by changing to spherical coords.

$$\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z+y^2z+z^3)\,dz\,dx\,dy$$

Solution E is the region $x^2+y^2+z^2 \le a^2$, so

$E = \{(\rho,\theta,\varphi) \mid 0 \le \rho \le a,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$, and

$x^2z+y^2z+z^3 = (x^2+y^2+z^2)z = \rho^2(\rho\cos\varphi)$

So the integral becomes

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{a} \rho^3\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$= \int_0^{\pi}\sin\varphi\cos\varphi\,d\varphi \int_0^{2\pi} d\theta \int_0^{a}\rho^5\,d\rho$$

$$= \left[\tfrac{1}{2}\sin^2\varphi\right]_0^{\pi}(2\pi)\left[\tfrac{1}{6}\rho^6\right]_0^{a}$$

$$= 0.$$