16.1 Vector fields

A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x,y) in $D \subset \mathbb{R}^2$ a two-dimensional vector $\vec{F}(x,y)$.

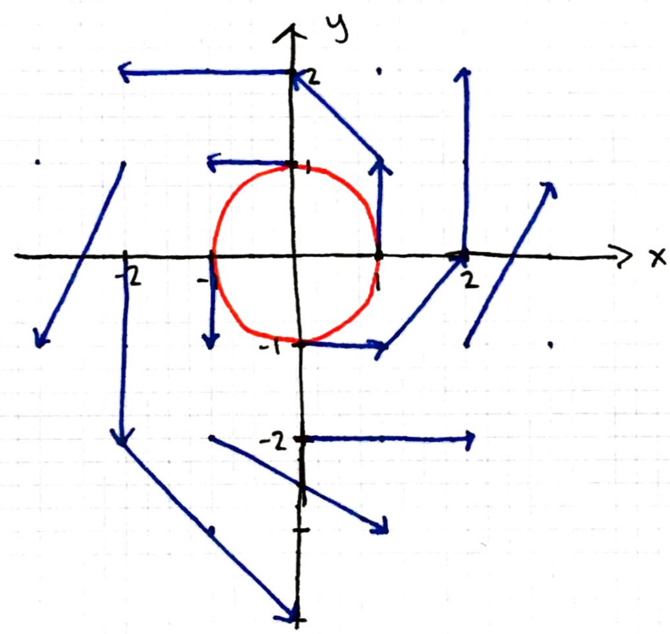
Similarly, if E is a subset of $\mathbb{R}^3$, a vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each (x,y,z) in E a vector $\vec{F}(x,y,z)$.

More generally, a vector field is a function $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$. That is, a function whose input is a vector in $\mathbb{R}^n$ and whose output is also a vector in $\mathbb{R}^n$.

Example 1 Sketch the vector field $\vec{F}(x,y) = \langle -y, x \rangle$.

Solution We pick some sample points

(x,y)	$\vec{F}(x,y)$	(x,y)	$\vec{F}(x,y)$	(x,y)	$\vec{F}(x,y)$	(x,y)	$\vec{F}(x,y)$	(x,y)	$\vec{F}(x,y)$
$(-2,-2)$	$\langle 2,-2 \rangle$	$(-1,-2)$	$\langle 2,-1 \rangle$	$(0,-2)$	$\langle 2,0 \rangle$	$(1,-2)$	$\langle 2,1 \rangle$	$(2,-2)$	$\langle 2,2 \rangle$
$(-2,-1)$	$\langle 1,-2 \rangle$	$(-1,-1)$	$\langle 1,-1 \rangle$	$(0,-1)$	$\langle 1,0 \rangle$	$(1,-1)$	$\langle 1,1 \rangle$	$(2,-1)$	$\langle 1,2 \rangle$
$(-2,0)$	$\langle 0,-2 \rangle$	$(-1,0)$	$\langle 0,-1 \rangle$	$(0,0)$	$\langle 0,0 \rangle$	$(1,0)$	$\langle 0,1 \rangle$	$(2,0)$	$\langle 0,2 \rangle$
$(-2,1)$	$\langle -1,-2 \rangle$	$(-1,1)$	$\langle -1,-1 \rangle$	$(0,1)$	$\langle -1,0 \rangle$	$(1,1)$	$\langle -1,1 \rangle$	$(2,1)$	$\langle -1,2 \rangle$
$(-2,2)$	$\langle -2,-2 \rangle$	$(-1,2)$	$\langle -2,-1 \rangle$	$(0,2)$	$\langle -2,0 \rangle$	$(1,2)$	$\langle -2,1 \rangle$	$(2,2)$	$\langle -2,2 \rangle$



Notice: Each vector is tangent to a circle centered at the origin.

Remark: We only need to draw a few representative vectors.

Notice that we can write $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$, where P, Q are real-valued functions. P and Q are called the component functions of $\vec{F}$.

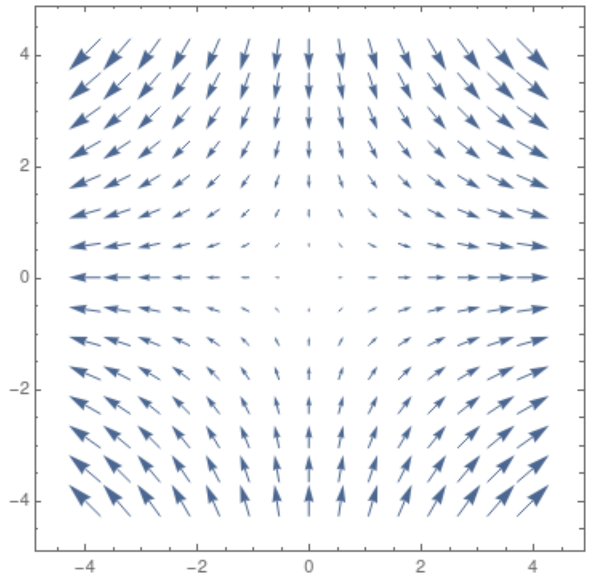
It gets pretty difficult to draw vector fields, especially in $\mathbb{R}^3$. So we generally use a computer to draw them. What's important is to be able to identify a vector field with a possible graph.

Another definition we need later:
A vector field is <u>Conservative</u> if it is the gradient of some scalar function. That is, if there exists some function f such that $\nabla f = \vec{F}$. In this situation, f is called a potential function for $\vec{F}$.

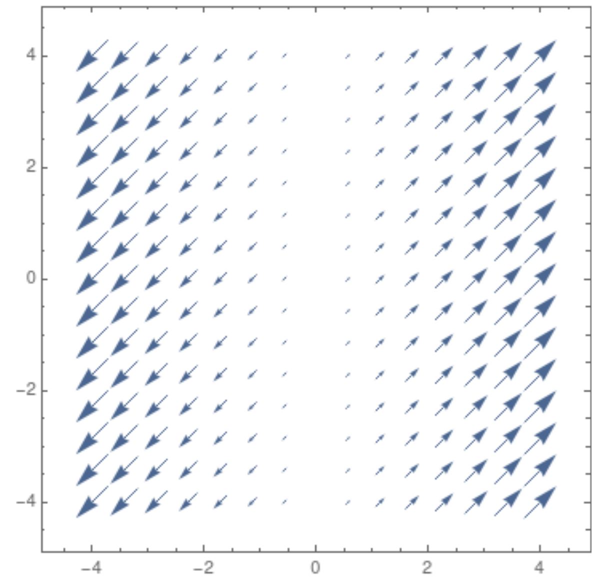
<u>Example 2</u> Find the gradient vector field of $f(x,y) = y\sin(xy)$.

<u>Solution</u> $\nabla f(x,y) = \langle y^2 \sin(xy), xy\cos(xy) + \sin(xy) \rangle$

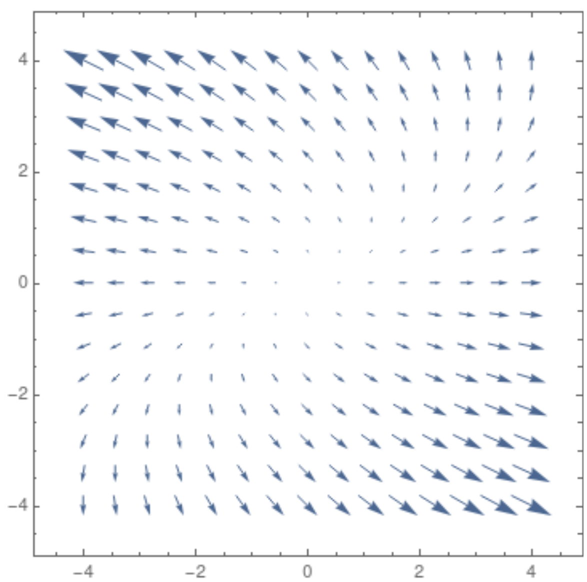
Match the vector fields **F** with the plots labeled I-IV.



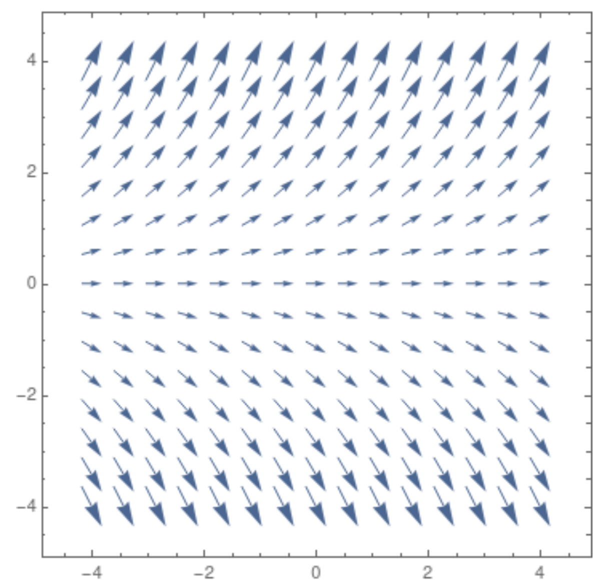
I



II



III



IV

(a) $\mathbf{F}(x,y) = < x - y, y >$
(b) $\mathbf{F}(x,y) = < 2, y >$
(c) $\mathbf{F}(x,y) = < x, -y >$
(d) $\mathbf{F}(x,y) = < x, x >$

For Graph I, notice that above the x-axis, the vectors point down and below the x-axis, they point up. This means that the y-component of **F** should have something that looks like -y. This leaves us with option (c).

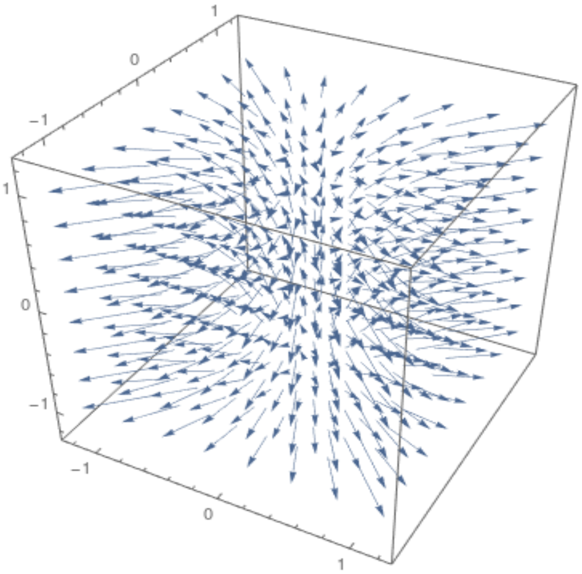
For Graph II, each of the vectors appears to have a slope of 1 or -1, so the components of **F** should be equal, giving us option (d).

For Graph III, notice that along the line y=x, the vectors appear to be vertical, suggesting that the x-component of **F** is 0 along this line. This agrees with option (a).

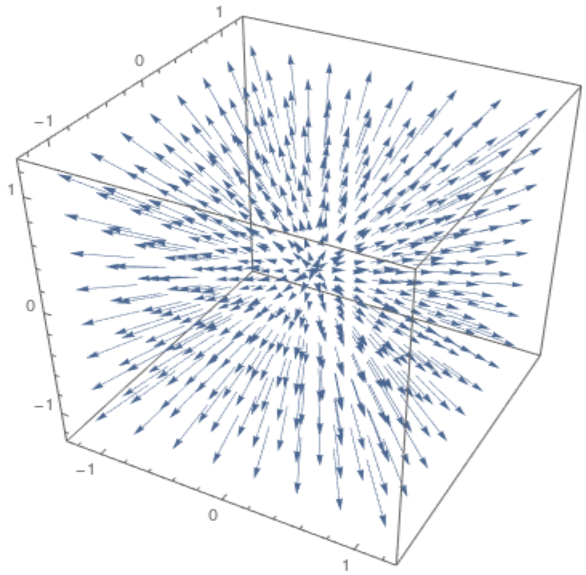
By process of elimination, Graph IV must be option (b). This lines up with the fact that all the vectors appear to have a constant x component.

Remark: This is the all the more level of reasoning you need for the homework and in general.

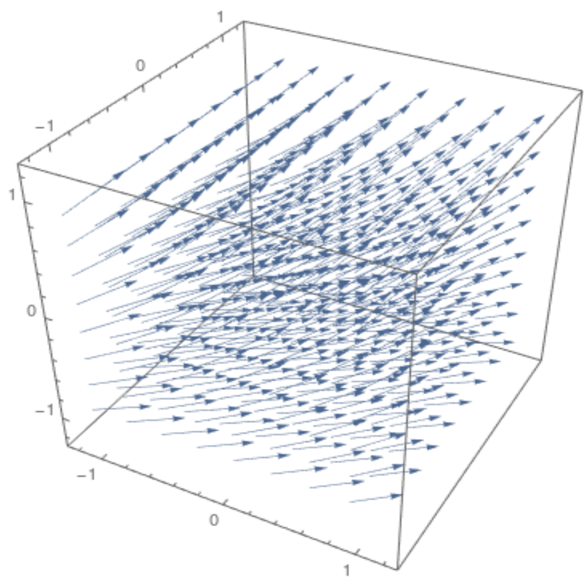
Match the vector fields **F** with the plots labeled I-IV.



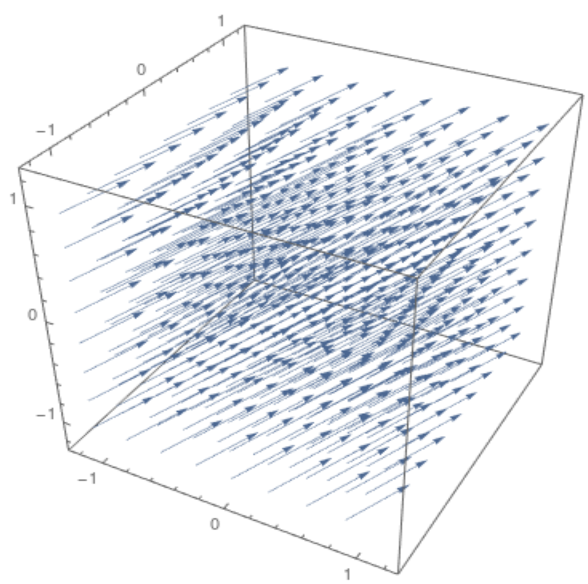
I



II



III



IV

(a) $\mathbf{F}(x,y) \ = < 2,3,1 >$
(b) $\mathbf{F}(x,y) \ = < x,y,0 >$
(c) $\mathbf{F}(x,y) \ =< x,y,z >$
(d) $\mathbf{F}(x,y) \ = < \ 1,2,z >$

In Graph I, it's hard to tell, but the z component of each of these vectors is 0, so the only vector field this works for is (b).

In Graph II, the vectors appear to explode radially, which makes option (c) the best choice for this one.

In Graph III, the z-direction of the vectors depends on the z-value, but they seem fixed in the x and y directions. This agrees with (d).

In Graph IV, all the vectors are fixed, which agrees with (a).