## 16.3 Line integrals, Fundamental Theorem

Recall, the fundamental theorem of Calculus says $\int_a^b F'(x)\,dx = F(b)-F(a)$, where $F'$ is continuous on $[a,b]$. Regarding $\nabla f$ as sort of the derivative of $f$, we have a Fundamental Theorem of line integrals.

Theorem Let $C$ be a smooth curve parametrized by $\vec{r}(t)$, $a \le t \le b$. Let $f$ be a differentiable multivariate function, whose gradient $\nabla f$ is continuous on $C$. Then

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).}$$

proof 
$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt \\ &= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt \quad \text{(chain rule)} \\ &= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \text{(Fundamental Thm of Calculus)}\end{aligned}$$

Remarks (1) We have proved the theorem in the case that $f$ is a function of three variables and $C$ is smooth. This can be readily generalized to a function $f$ of any number of variables and allow $C$ to be just piecewise-smooth.

(2) The theorem says we can evaluate the line integral of a conservative vector field ($\vec{F} = \nabla f$) by knowing the value of $f$ at the end points.

(3) This means that if $\vec{F} = \nabla f$, then

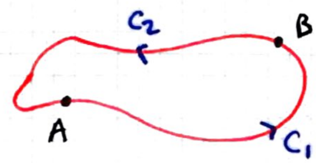
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

for any two smooth curves with the same initial and terminal points. In this case, we say $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

The theorem above has a consequence for closed curves; that is, a curve where the initial and terminal points coincide. Again we assume $\vec{F}$ is a continuous vector field with domain $D$.

Theorem $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in $D$ if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path $C$ in $D$.

<u>proof</u> Suppose $\int_C \vec{F}\cdot d\vec{r}$ is independent of path. We want to show $\int_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve $C$ in $D$. So let $C$ be a closed curve in $D$. Pick points $A$ and $B$ on $C$. Then we can think of $C$ as a path $C_1$ from $A$ to $B$ followed by a path $C_2$ from $B$ to $A$.




Then, $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r} = 0$

since $C_1$ and $-C_2$ have the same initial and terminal points.

Conversely, suppose $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed curve $C$ in $D$. Let $C_1$ and $C_2$ be any two paths from $A$ to $B$. Then $C_1$ followed by $-C_2$ is a closed curve, say $C$. Then

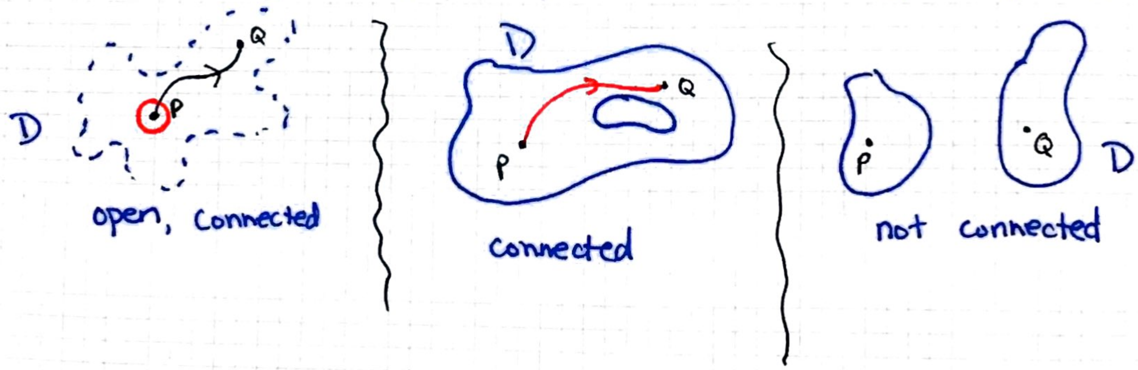
$$0 = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}.$$

Thus $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$.

It turns out that the only vector fields that are independent of path are conservative:

<u>Theorem</u> Suppose $\vec{F}$ is a continuous vector field on an open connected region $D$. If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in $D$, then $\vec{F}$ is a conservative vector field in $D$.

Recall that $D$ is <u>open</u> if for any point $P$ in $D$ there is a disk centered at $P$ lying entirely in $D$. And $D$ <u>connected</u> means that any two points in $D$, $P, Q$, can be joined by a path.



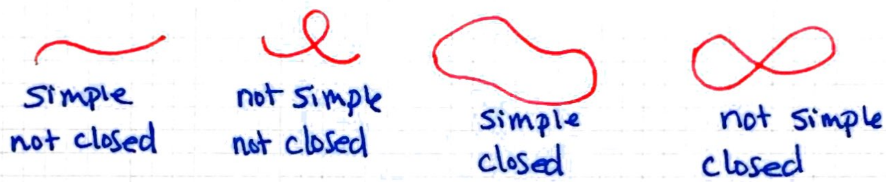
__Theorem__ If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a conservative vector field, where $P$ and $Q$ have continuous first-order partial derivatives on a domain (open connected) $D$, then throughout $D$, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

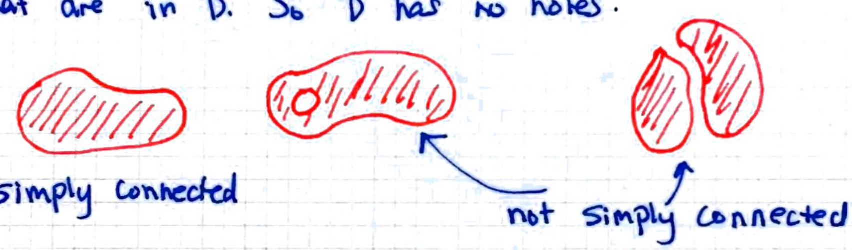
We're really interested in a converse to the above theorem. That is, given a continuous vector field $\vec{F}$, when can we say $\vec{F}$ is conservative?

A few definitions:

A __simple curve__ is one that doesn't intersect itself.




A __simply-connected__ region in the plane is a connected region $D$ such that every simple closed curve in $D$ encloses only points that are in $D$. So $D$ has no holes.




__Theorem__ $\vec{F} = \langle P, Q \rangle$ a vector field on a simply connected domain $D$. Suppose $P$ and $Q$ have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D.$$

Then $\vec{F}$ is conservative.

__Example 1__ Determine whether the vector field $\vec{F}$ is conservative.

(a) $\vec{F}(x,y) = \langle x-y, x-2 \rangle$

(b) $\vec{F}(x,y) = \langle 3+2xy, x^2-3y^2 \rangle$.

**Solution** (a) $P(x,y) = x-y \Rightarrow \frac{\partial P}{\partial y} = -1$ while $Q(x,y) = x-2 \Rightarrow \frac{\partial Q}{\partial x} = 1$. So $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$. So $\vec{F}$ is not conservative.

(b) $P(x,y) = 3+2xy \Rightarrow \frac{\partial P}{\partial y} = 2x$ and $Q(x,y) = x^2-3y^2 \Rightarrow \frac{\partial Q}{\partial x} = 2x$.
So $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Also, the domain of $\vec{F}$ is $D = \mathbb{R}^2$, which is open and simply connected. Thus $\vec{F}$ is conservative

How do we find the potential function $f$ ($\vec{F} = \nabla f$)? We use sort of "partial integration."

**Example 2** Determine whether $\vec{F}$ is conservative. If so, find $f$ such that $\nabla f = \vec{F}$.

$$\vec{F} = \langle y^2 - 2x, 2xy \rangle$$

**Solution** $P(x,y) = y^2 - 2x \Rightarrow \frac{\partial P}{\partial y} = 2y$ and $\frac{\partial Q}{\partial x} = 2y$. Again the domain of $\vec{F}$ is $\mathbb{R}^2$, so $\vec{F}$ is conservative. So $\exists f$ such that $\nabla f = \vec{F}$. That is,

$$f_x = y^2 - 2x \quad \text{and} \quad f_y = 2xy. \qquad (*)$$

Now $f_x = y^2 - 2x \Rightarrow f(x,y) = xy^2 - x^2 + g(y)$ (integrating wrt $x$).
Thus $f_y = 2xy + g'(y)$ (taking partial wrt $y$).
But from $(*)$ we know that $f_y = 2xy$. Thus $g'(y) = 0$. So $g(y) = K$, some constant. So $f(x,y) = xy^2 - x^2 + K$. Since we are just looking for a function $f$, we can pick $K=0$, so $f(x,y) = xy^2 - x^2$ satisfies $\nabla f = \vec{F}$.

**Example 3** Determine whether $\vec{F}$ is conservative, if so find $f$ w/ $\nabla f = \vec{F}$.

$$F(x,y) = \langle ye^x, e^x + e^y \rangle$$

**Solution** $\frac{\partial}{\partial y}(ye^x) = e^x = \frac{\partial}{\partial x}(e^x + e^y)$ and the domain of $\vec{F}$ is $\mathbb{R}^2$, so $\vec{F}$ is conservative.

$$f_x = ye^x \quad \text{and} \quad f_y = e^x + e^y.$$

So $f(x,y) = ye^x + g(y)$, and $f_y = e^x + g'(y) = e^x + e^y \Rightarrow g'(y) = e^y$.
So $g(y) = e^y + K$. So $f(x,y) = ye^x + e^y + K$. Again, we can choose $K=0$.

Example 4 Find a function $f$ such that $\nabla f = \vec{F}$, then evaluate $\int_C \vec{F} \cdot d\vec{r}$. $\vec{F} = \langle 3 + 2xy^2, 2x^2y \rangle$ and $C$ is the arc of the hyperbola $y = 1/x$ from $(1,1)$ to $(4, 1/4)$.

Solution

$f_x = 3 + 2xy^2$ and $f_y = 2x^2y$.

So $f(x,y) = 3x + x^2y^2 + g(y) \Rightarrow f_y = 2x^2y + g'(y) = 2x^2y \Rightarrow g(y) = K$.

Thus $f(x,y) = 3x + x^2y^2$ satisfies $\nabla f = \vec{F}$.

Now $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(4, 1/4) - f(1,1) = (12+1) - (3+1) = 9$.