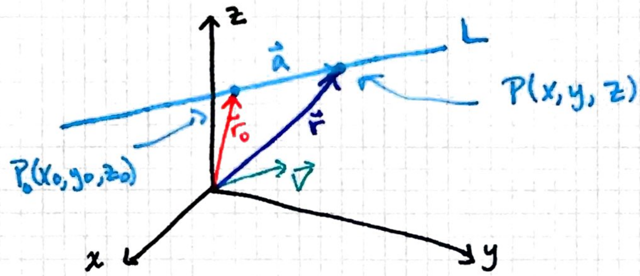
12.5 Equations of lines and planes.

In the xy-plane, a line is given by point + direction (slope). Same for $\mathbb{R}^3$.



r_0, r position vectors for P_0, P, resp.

$r = r_0 + a$

$\vec{a}$ parallel to $\vec{v}$ $\Rightarrow$ $\vec{a} = t\vec{v}$. Gives

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}} \qquad (*)$$

vector equation for a line with parameter t. If we consider the components of the vectors in (*), we get parametric eqns of the line:

Parametric eqns for a line through (x_0, y_0, z_0) parallel to the vector $\langle a, b, c \rangle$: $x = x_0 + at$ $y = y_0 + bt$ $z = z_0 + ct$

Example 1 (a) Find ~~the~~ vector and parametric eqns for the line through $(1,2,3)$ and parallel to $\langle 3, 2, -1 \rangle$.

Solution $\vec{r} = \langle 1, 2, 3 \rangle + t \langle 3, 2, -1 \rangle$ (vector)

$x = 1 + 3t, \; y = 2 + 2t, \; z = 3 - t$ (parametric)

(b) Find ~~the~~ vector/parametric eqns for the line through $(4, 2, 3)$ and perpendicular to both $\langle -1, 0, 2 \rangle$ and $\langle 1, 1, 1 \rangle$

Solution $\vec{v} = \langle -1, 0, 2 \rangle \times \langle 1, 1, 1 \rangle = \langle -2, 3, -1 \rangle$

$\vec{r} = \langle 4, 2, 3 \rangle + t \langle -2, 3, -1 \rangle$ (vector)

$x = 4 - 2t, \; y = 2 + 3t, \; z = 3 - t$ (parametric

Note These equations are not unique.

Def If $\vec{v} = \langle a, b, c \rangle$ describes direction of a line, L, then a, b, c are called the <u>direction numbers</u> of L.

Note Any triple proportional to a, b, c could be used.

We can solve the parametric eqns of a line for t to obtain

$$t = \frac{x - x_0}{a}, \quad t = \frac{y - y_0}{b}, \quad t = \frac{z - z_0}{c}, \quad \text{giving}$$

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

These are called ~~the~~ <u>symmetric equations</u> of L.

Remark We can still find symmetric eqns if one of a, b or c is 0.

e.g. If $a = 0$, we can write

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example 2 Find symmetric eqns for the line L through the points A(-8, 1, 4) and B(3, -2, 4).

Solution The vector $\vec{v} = \vec{AB}$ is parallel to L.

$$\vec{v} = \langle 3, -2, 4 \rangle - \langle -8, 1, 4 \rangle = \langle 11, -3, 0 \rangle.$$

So the direction numbers are $a = 11$, $b = -3$, $c = 0$.

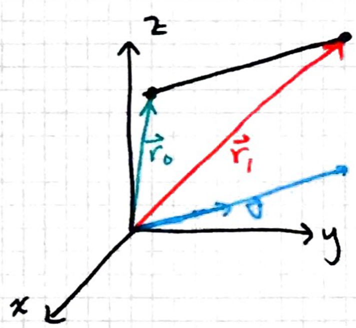
Take $(x_0, y_0, z_0) = (-8, 1, 4)$. Then

$$\frac{x+8}{11} = \frac{y-1}{-3}, \quad z = 4.$$

How to describe a line segment?

Consider ~~para~~ vector eqns

$$\vec{r} = \vec{r}_0 + t\vec{v}.$$



If $t = 0$, then $r = r_0$. If we take $v = \vec{r}_1 - \vec{r}_0$, then when $t = 1$, $\vec{r} = r_0 + (\vec{r}_1 - \vec{r}_0) = \vec{r}_1$.

Thus, the vector equation of the segment from $\vec{r}_0$ to $\vec{r}_1$ is

$$\boxed{\vec{r}(t) = (1-t)r_0 + t\vec{r}_1} \quad \underline{0 \le t \le 1}$$

$$\left(\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) = (1-t)\vec{r}_0 + t\vec{r}_1\right)$$

Example 3 Find eqn of segment from A to B in Ex2.

Soln $\vec{r}_0 = \langle -8, 1, 4\rangle$, $\vec{r}_1 = \langle 3, -2, 4\rangle$

$$\vec{r}(t) = (1-t)\langle -8, 1, 4\rangle + t\langle 3, -2, 4\rangle, \quad 0 \le t \le 1.$$

Could also write $\vec{r}(t) = \langle -8 + 11t, 1 - 3t, 4\rangle, \quad 0 \le t \le 1.$

In $\mathbb{R}^3$, it's possible to have lines that do not intersect and are not parallel. These are called skew lines.

Example 4 Show that the lines L_1 and L_2 are skew, where

L_1: $x = 3 + 2t$, $y = 4 - t$, $z = 1 + 3t$

L_2: $x = 1 + 4s$, $y = 3 - 2s$, $z = 4 + 5s$.

Solution The direction vectors for L_1, L_2 are $\vec{v}_1 = \langle 2, -1, 3\rangle$, $\vec{v}_2 = \langle 4, -2, 5\rangle$, which are not proportional $\Rightarrow$ not parallel. If L_1 intersects L_2, then there exists t, s satisfying

$$3 + 2t = 1 + 4s \quad (1)$$
$$4 - t = 3 - 2s \quad (2)$$
$$4 + 5s = 1 + 3t. \quad (3)$$

(2) gives $t = 1 + 2s$, subbing in (1) gives

$$3 + 2(1 + 2s) = 1 + 4s$$
$$5 + 4s = 1 + 4s$$
$$5 = 1. \quad \lightning$$

Planes

A plane is described by a point and a normal vector, $\vec{n}$, that is a vector orthogonal to the plane. Specify P_0, and let P be an arbitrary vector in the plane. Let $\vec{r}_0$, $\vec{r}$ be their position vectors. Then $\overrightarrow{P_0P} = \vec{r} - \vec{r}_0$, and

$$\boxed{\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0} \quad (4) \quad (\vec{n} \text{ is orthogonal to } \vec{r} - \vec{r}_0)$$

or,

$$\boxed{\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0} \quad (5)$$

These are called vector equations for a plane.

Let $\vec{n} = \langle a, b, c\rangle$, $\vec{r} = \langle x, y, z\rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$.

Then (4) becomes

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

or $$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0.$$

So a <u>scalar eqn</u> of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$. is

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0.}$$

We could multiply everything out, collect constants to get a <u>linear eqn</u> in x, y, z for the plane

$$\boxed{ax + by + cz + d = 0.}$$

<u>Example 5</u> Find an eqn of the plane
(a) through $(5, 3, 5)$ with normal vector $\vec{n} = \langle 2, 1, -1\rangle$
(b) through the points $P(2,1,2)$, $Q(3,-8,6)$, $R(-2,-3,1)$

<u>Solution</u> (a) $2(x-5) + 1(y-3) + (-1)(z-5) = 0.$

(b) $\vec{a} = \vec{PQ} = \langle 1, -9, 4\rangle$, $\vec{b} = \vec{PR} = \langle -4, -4, -1\rangle$

both lie in the plane, so a normal vector is

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -9 & 4 \\ -4 & -4 & -1 \end{vmatrix} = \langle 25, -15, -40\rangle. = \langle 5, -3, -8\rangle.$$

So, an eqn of the plane is

$$5(x-2) + (-3)(y-1) - 8(z-2) = 0$$

or $$5x - 3y - 8z = -9.$$

<u>Def</u> Two planes are <u>parallel</u> if their normal vectors are parallel. If the two planes are not parallel, the angle between them is the acute angle between their normal vectors.

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}.$$

Example 6 Find the angle between the planes and the line of intersection.

$$3x - 2y + z = 1, \quad 2x + y - 3z = 3.$$

Solution The planes have normal vectors $\vec{n}_1 = \langle 3, -2, 1 \rangle$ and $\vec{n}_2 = \langle 2, 1, -3 \rangle$. So

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{(3)(2)+(-2)(1)+(1)(-3)}{\sqrt{9+4+1}\ \sqrt{4+1+9}} = \frac{1}{2\sqrt{14}}$$

So $\theta = \cos^{-1}\left(\frac{1}{2\sqrt{14}}\right) \approx 1.437$

To find eqn of line of intersection L, we need a point on L. We can pick (arbitrarily) $z=0$. This gives

$$\begin{cases} 3x - 2y = 1 \\ 2x + y = 3 \end{cases}$$

Solving, gives $x = y = 1$. Thus $(1,1,0)$ is a point on L.
L lives in both planes $\Rightarrow$ perpendicular to both $\vec{n}_1$ and $\vec{n}_2$.
So take

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 5, 11, 7 \rangle.$$ Then vector eqn is

$$\vec{r} = \langle 1, 1, 0 \rangle + t\langle 5, 11, 7 \rangle.$$

12.6 Cylinders and quadric surfaces.

Cylinders A cylinder is a surface consisting of all lines (called rulings) parallel to a given line passing through some plane curve.

Middle school example: plane curve is a circle in the xy-plane and the given line is z-axis.

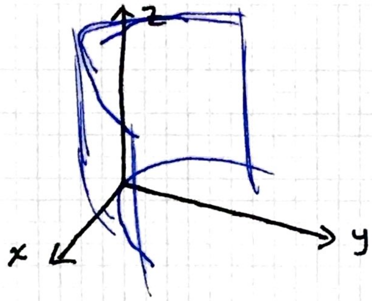
Example 7 Sketch the surfaces

(a) $y = x^2$ parabolic cylinder

(b) $y = e^x$

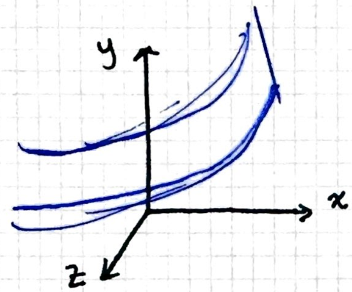
(c) $\frac{x^2}{9} + \frac{y^2}{4} = 1.$ elliptic cylinder

Solution (a)

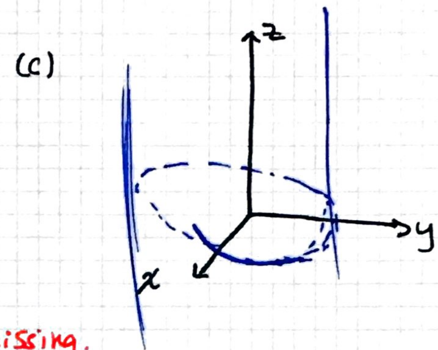


$y = x^2$

(b)



(c)



Notice Always one variable missing.

Quadric surfaces

A quadric surface is a graph of a second degree eqn in x, y, z.

General eqn: $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$,
$A, B, \dots, J$ constants. By translation / rotation / messy algebra, can reduce to one of two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

(*) Counterparts to conic sections in the plane.

Example 8 Use traces (cross-sections) to sketch the quadric surface.

(a) $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$

(b) $z = \frac{x^2}{9} + \frac{y^2}{4}$

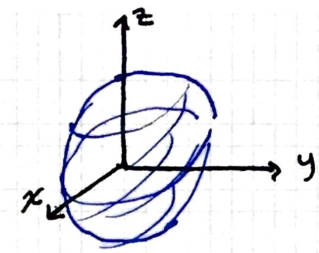
(c) $z = \frac{x^2}{9} - \frac{y^2}{4}$

(d) $\frac{x^2}{9} + \frac{y^2}{4} - z^2 = 1$.

Solution (a) When $z = 0$, we get $\frac{x^2}{9} + \frac{y^2}{4} = 1$, ellipse.
For $z = k$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 - k^2 \quad \text{is an ellipse for } |k| \le 1.$$

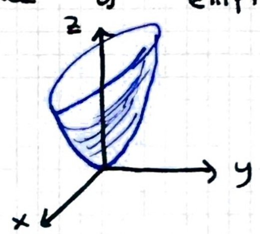
Similarly, vertical traces are also ellipses. Graph is ellipsoid.



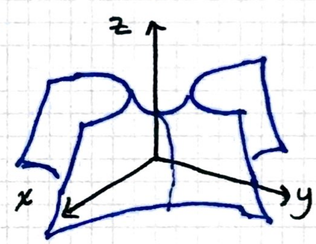
(b) Put $x=0$, get $z = y^2/4$: parabola. Put $\cancel{z} = k$, get $z = \frac{y^2}{4} + \frac{k^2}{9}$, gives upward parabola for any plane parallel to yz-plane. Similar for $y=k$. For $z=k$, get

$$\frac{y^2}{4} + \frac{x^2}{9} = k \qquad \text{ellipses}$$

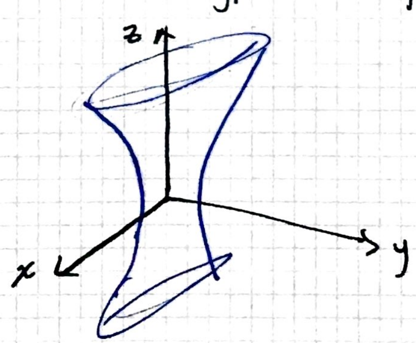
So graph of surface is elliptic paraboloid.



(c) For $x=k$, get $\cup$ parabolas, for $y=k$ get $\cap$ parabolas, for $z=k$ get $\frac{1}{4}y^2 - \frac{1}{9}x^2 = k$: hyperbolas. Result: hyperbolic paraboloid.



(d) For $z=k$, get ellipse. $y=0$: $\frac{x^2}{9} - z^2 = 1$, $x=0$: $\frac{y^2}{4} - z^2 = 1$ are hyperbolas. Result: hyperboloid of one sheet.



Example 9 Classify $4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$

Solution Complete the square!

$$4(x^2 - 6x + 9) + (y^2 - 8y + 16) + (z^2 + 4z + 4) + 55 = 56$$

$$4(x-3)^2 + (y-4)^2 + (z+2)^2 = 1$$

ellipsoid

Table 1 pg 837 very useful.