16.5 Curl and divergence.

Before introducing curl and divergence, we need to introduce the operator "del", ∇. It takes a differentiable function of multiple variables and produces its gradient. So, $\nabla f = \text{grad } f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$.

If the notion of an operator is confusing, you already know multiple examples. d/dx is an operator that takes a function $f(x)$ and gives $f'(x)$. A matrix takes a vector and gives another vector via left multiplication.

Curl

Let $\vec{F} = \langle P, Q, R \rangle$ be a vector field on $\mathbb{R}^3$, and assume the partial derivatives of P, Q, R all exist. Then we define the curl of $\vec{F}$ on $\mathbb{R}^3$ by

$$\boxed{\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{j}.}$$

Note that if we write the del operator as $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$, then we can write the curl of $\vec{F}$ as the formal cross product

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Example 1 Find the curl of $\vec{F}(x,y,z) = x^3yz^2\hat{j} + y^4z^3\hat{k}$

Solution

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^3yz^2 & y^4z^3 \end{vmatrix} =$$

$$= \left[\frac{\partial}{\partial y}(y^4z^3) - \frac{\partial}{\partial z}(x^3yz^2)\right]\hat{i} - \left[\frac{\partial}{\partial x}(y^4z^3) - 0\right]\hat{j} + \left[\frac{\partial}{\partial x}(x^3yz^2) - 0\right]\hat{k}$$

$$= \left[4y^3z^3 - 2x^3yz\right]\hat{i} - 0\hat{j} + 3x^2yz^2\hat{k}$$

Theorem If f is a function of 3 variables with continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = 0.$$

Proof We write out $\nabla \times (\nabla f)$ and use Clairaut's Theorem.

Note Since a conservative vector field $\vec{F}$ has some f such that $\vec{F} = \nabla f$, the theorem says that if $\vec{F}$ is conservative, then $\text{curl}\,\vec{F} = 0$. For example, the vector field in Example 1 is not conservative because $\text{curl}\,\vec{F} \neq 0$.

The converse is not true in general, but there is a case where it is.

Theorem If $\vec{F}$ is a vector field defined on an open simply-connected region in $\mathbb{R}^3$, and if the component functions of $\vec{F}$ all have continuous partial derivatives and $\text{curl}\,\vec{F} = 0$, then $\vec{F}$ is conservative.

Remark If we write $\vec{F}(x,y) = F(x,y,0)$, then the theorem for $\mathbb{R}^2$ is seen as a special case of this one.

Example 2 Determine whether $\vec{F}$ is conservative; if it is, find f such that $\vec{F} = \nabla f$, where $\vec{F}(x,y,z) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$.

Solution Notice that $\vec{F}$ is defined on all of $\mathbb{R}^3$, so we need to check $\text{curl}\,\vec{F} = 0$. Indeed,

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \left[xyze^{yz} + xe^{yz} - (xyze^{yz} + xe^{yz})\right]\hat{\imath} - \left[ye^{yz} - ye^{yz}\right]\hat{\jmath} + \left[ze^{yz} - ze^{yz}\right]\hat{k} = 0.$$

Now $f_x = e^{yz}$, $f_y = xze^{yz}$, $f_z = xye^{yz}$. So,

$f(x,y,z) = xe^{yz} + g(y,z) \Rightarrow f_y = xze^{yz} + g_y(y,z)$. Comparing with f_y above shows $g_y = 0 \Rightarrow g(y,z) = h(z)$. Now, $f(x,y,z) = xe^{yz} + h(z) \Rightarrow f_z = xye^{yz} + h'(z)$. Again comparing above, $h'(z) = 0 \Rightarrow h(z) = K$. So $f(x,y,z) = xe^{yz} + K$.

Why the name curl? Given a vector field $\vec{F}$ and a point (x,y,z), $\text{curl}\,\vec{F}(x,y,z)$ is a vector that gives the axis of rotation of the vector field near (x,y,z). $|\text{curl}\,\vec{F}(x,y,z)|$ represents how quickly particles rotate about the axis.

Divergence

If $\vec{F} = P\hat{\imath} + Q\hat{k} + R\hat{\jmath}$, we define the divergence of $\vec{F}$

$$\boxed{\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Note While $\text{curl}\,\vec{F}$ is a vector field, $\text{div}\,\vec{F}$ is a scalar field.

Example 3 Find div $\vec{F}$, where $\vec{F}(x,y,z) = x^3yz^2\,\hat{j} + y^4z^3\,\hat{k}$.

Solution
$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(x^3yz^2) + \frac{\partial}{\partial z}(y^4z^3)$$
$$= x^3z^2 + 3y^4z^2.$$

In the context of fluid flow, div $\vec{F}$ represents the net rate of change of the mass of fluid flowing from a point. A positive divergence says the net flow is outward, and a negative divergence says the net flow is into the point.

Example 4 Let f be a scalar field, $\vec{F}$ a vector field. State whether each expression is meaningful. If it is, state whether it is a scalar or vector field.

(a) grad(div $\vec{F}$)
(b) curl(curl($\vec{F}$))
(c) (grad f) $\times$ (div $\vec{F}$)

Solution

(a) meaningful — div $\vec{F}$ is a scalar field, so we can take grad(div $\vec{F}$), which produces a vector field.

(b) meaningful — curl $\vec{F}$ is a vector field, so we can take curl(curl $\vec{F}$). The result is a vector field.

(c) meaningless — div $\vec{F}$ is a scalar field, so we cannot take the cross product.

Example 5 Assuming the appropriate partial derivatives exist and are continuous, prove that $\operatorname{div}(f\vec{F}) = f\operatorname{div}\vec{F} + \vec{F}\cdot\nabla f$, where $f\vec{F}(x,y,z) = f(x,y,z)\vec{F}(x,y,z)$.

Solution Write $\vec{F} = \langle P, Q, R\rangle$. Then

$$\begin{aligned}
\operatorname{div}(f\vec{F}) &= \operatorname{div}(\langle fP, fQ, fR\rangle) \\
&= \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR) \\
&= P\frac{\partial f}{\partial x} + f\frac{\partial P}{\partial x} + Q\frac{\partial f}{\partial y} + f\frac{\partial Q}{\partial y} + R\frac{\partial f}{\partial z} + f\frac{\partial R}{\partial z} \\
&= f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) + \left(P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z}\right) \\
&= f\operatorname{div}\vec{F} + \vec{F}\cdot\nabla f.
\end{aligned}$$

Vector form of Green's Theorem

With the notation and assumptions from Green's Theorem, we can regard $\vec{F} = \langle P, Q \rangle$ as a vector field on $\mathbb{R}^3$ with z-component 0. Then

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

Thus $(\text{Curl } \vec{F}) \cdot \hat{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} \cdot \hat{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, and we can write

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot k \, dA.$$