16.6 Parametric surfaces

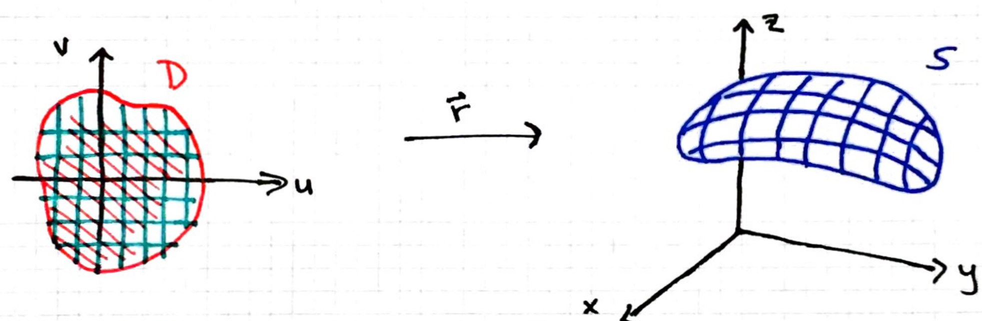
We know how to parametrize (some) space curves from earlier lessons. We do this by setting $x=f(t)$, $y=g(t)$, $z=h(t)$. We want to do something similar for surfaces.

Notice that to parametrize any space curve you need only one parameter (t). This is because any curve is 1-dimensional: you can go forward or backward on the curve. To describe a surface we'll need two parameters. We'll usually call them u, v.

We can do this by using a vector function $\vec{r}(u,v)$ defined on some region D in the uv-plane:

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle.$$

Or by parametric equations $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$.



We call S a parametric surface.

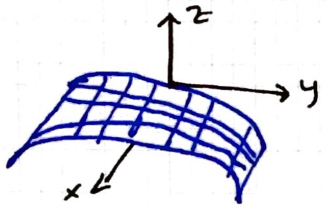
One thing to notice is that the lines $u=u_0$ and $v=v_0$ get sent to curves on the surface S. These curves are called the "grid curves" of S.

Example 1 Identify and sketch the surface given by

$$\vec{r}(u,v) = \langle u,\ 3\cos v,\ \sin v \rangle, \quad 0 \le u \le 2, \quad 0 \le v \le \pi$$

Solution Parametrically, we have $x=u$, $y=3\cos v$, $z=\sin v$.

Notice that $\frac{y^2}{9} + z^2 = 1$, so the cross-sections in the yz-plane are half-ellipses (since $0 \le v \le \pi$). Then $x=u$ ranges from 0 to 2. So this is the top half of an elliptic cylinder along the x-axis with height 2.



Example 2 Identify the surface given by

$$\vec{r}(s,t) = \langle 3s \cdot \sin t,\ 5s \cos t,\ s^2 \rangle.$$

Solution Here we have $x = 3s \cdot \sin t$, $y = 5s \cdot \cos t$, $z = s^2$. So,

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = s^2 = z,$$ which is an elliptic paraboloid opening to the positive z-axis.

It's useful to be able to identify the grid curves of a parametric surface to have a better understanding of the surface.

Example 3 Identify the grid curves of $\vec{r}$ and match it with one of the surfaces given on the next page.

$$\vec{r}(u,v) = \langle \sin u,\ \cos u \sin v,\ \sin v \rangle$$

Solution

For $v = v_0$, $\sin v_0 = c$ is a constant; so $\vec{r}(u, v_0) = \langle \sin u, c \cdot \cos u, c \rangle$, which is an ellipse in the plane $z = c$. If $u = u_0$, then $\sin u_0 = a$, $\cos u_0 = b$ are constants. So $\vec{r}(u_0, v) = \langle a, b\sin v, \sin v \rangle \Rightarrow z = by$ are lines in the -plane. $x = a$. This corresponds to graph IV.

Example 4 Find a vector function for the plane through P_0, containing vectors $\vec{a}, \vec{b}$.

Solution Let $\vec{r}_0$ be the position vector of P_0. Then for any point P in the plane we can get from P_0 to P by moving some amount in the $\vec{a}$ direction followed by some amount in the $\vec{b}$ direction (The parallelogram law). That is

$$\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$$

Example 5 Find a parametric representation of the sphere $x^2+y^2+z^2=4$.

Solution We use spherical coordinates.

$$\vec{r}(\varphi,\theta) = \langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi \rangle, \quad 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi$$

Example 6 Parametrize the cone $z^2 = 4x^2+4y^2$ for $z \ge 0$.

Solution We can do this in two different ways: since $z \ge 0$, we can solve the equation of the cone for z to get $z = \sqrt{4x^2+y^2}$.

Then $\vec{r}(x,y) = \langle x, y, \sqrt{4x^2+y^2} \rangle$. Or we can use cylindrical coordinates:

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r \rangle, \quad r \ge 0, \quad 0 \le \theta \le 2\pi,$$

where $z = \sqrt{4r^2} = 2r$.

Remark The first parametrization of the cone in Example 6 is as a graph. In general, if $z = f(x,y)$ describes a surface we want to parametrize, we can pick $x=x$, $y=y$ to get

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle.$$

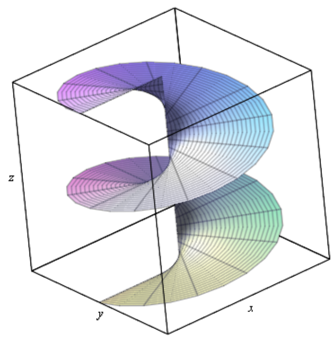
What's nice about cylindrical coords in Example 6, though, is that if we want to restrict $0 \le z \le 2$, all we have to do is restrict $0 \le r \le 1$. The bounds for x and y in the graph parametrization aren't as clear. So although parametrizing as a graph may be easy to write the equations, the bounds may be easier with another parametrization

Ex 7 Parametrize the part of the ellipsoid $x^2+2y^2+3z^2=1$ to the left of the xz-plane.

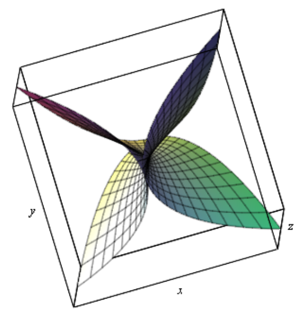
Solution Here $y \le 0$, so $y = -\sqrt{\frac{1}{2}(1-x^2-3z^2)}$, and we can write

$$\vec{r}(x,z) = \langle x, -\sqrt{\tfrac{1}{2}(1-x^2-3z^2)}, z \rangle.$$

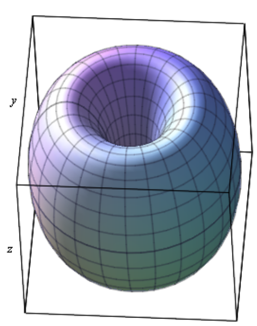
There are at least two other parametrizations, but this one is the simplest/quickest.



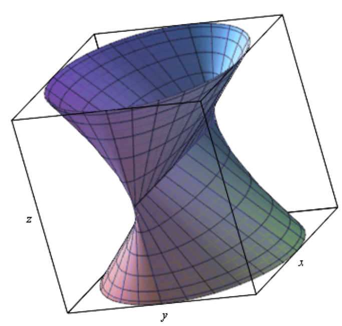
(a) Graph I



(b) Graph II



(a) Graph III



(b) Graph IV