16.6 Areas of parametric surfaces

Given a parametric surface S with vector function

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle,$$

we want to find an equation of the tangent plane at $\vec{r}(a,b)$.
Looking at the grid curves we find that $\vec{r}_u$ and $\vec{r}_v$ describe tangent lines, which means that $\vec{r}_u \times \vec{r}_v$ is a normal vector to the tangent plane, provided that $\vec{r}_u \times \vec{r}_v \neq 0$. (When $\vec{r}_u \times \vec{r}_v \neq 0$, we say that S is smooth)

Example 1 Find an equation of the tangent plane to

$$x = u^2 + 1,\quad y = v^3 + 1,\quad z = u + v \quad \text{at} \quad (5,2,3).$$

Solution

Here $\vec{r}(u,v) = \langle u^2+1,\ v^3+1,\ u+v\rangle$, so $\vec{r}_u = \langle 2u, 0, 1\rangle$ and $\vec{r}_v = \langle 0, 3v^2, 1\rangle$.
And the point $(5,2,3)$ corresponds to $(u,v) = (2,1)$. So we want to find $\langle 4,0,1\rangle \times \langle 0,3,1\rangle = \langle -3,-4,12\rangle$, which is a normal vector to the tangent plane. So an equation is

$$-3(x-5) - 4(y-2) + 12(z-3) = 0$$

or

$$3x + 4y - 12z = -13$$

Surface Area

If a smooth parametric surface S is given by

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle, \quad (u,v) \in D$$

and S is covered just once as (u,v) ranges throughout D, then the surface area of S is

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$$

Example 2 Find the area of the part of the surface of the paraboloid $y = x^2 + z^2$ that lies within the cylinder $x^2 + z^2 = 16$.

Solution We can parametrize the surface by

$$\vec{r}(x,z) = \langle x,\ x^2+z^2,\ z\rangle \quad \text{with} \quad 0 \le x^2 + z^2 \le 16.$$

Now $\vec{r}_x = \langle 1, 2x, 0\rangle$, $\vec{r}_z = \langle 0, 2z, 1\rangle$, so $\vec{r}_x \times \vec{r}_z = \langle 2x, -1, 2z\rangle$.

Then

$$A(S) = \iint_{x^2+z^2 \le 16} \sqrt{1+4x^2+4z^2}\, dA$$

$$= \int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\; r\, dr\, d\theta = \frac{\pi}{6}\left(65^{3/2}-1\right)$$

__Notice__ If S is a graph, ie. $z=f(x,y)$, then S can be parametrized by $\vec{r}(x,y) = \langle x, y, f(x,y)\rangle$, and

$$\vec{r}_x = \langle 1, 0, \tfrac{\partial f}{\partial x}\rangle, \quad \vec{r}_y = \langle 0, 1, \tfrac{\partial f}{\partial y}\rangle \Rightarrow \vec{r}_x \times \vec{r}_y = \langle -\tfrac{\partial f}{\partial x}, -\tfrac{\partial f}{\partial y}, 1\rangle,$$

so that

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+\left(\tfrac{\partial f}{\partial x}\right)^2 + \left(\tfrac{\partial f}{\partial y}\right)^2}$$

and today's integral describing surface area of S coincides with the one from chapter 15. As Example 2 shows, we have similar formulas if S is given by $g(x,z)$ or $h(y,z)$.

__Example 3__ Find the area of the surface $x = z^2+y$ that lies between the planes $y=0$, $y=2$, $z=0$ and $z=2$.

__Solution__ $\vec{r}(y,z) = \langle z^2+y, y, z\rangle$, $0 \le y \le 2$, $0 \le z \le 2$. So $\vec{r}_y = \langle 1,1,0\rangle$, $\vec{r}_z = \langle 2z, 0, 1\rangle$, and $\vec{r}_y \times \vec{r}_z = \langle 1, -1, -2z\rangle$

and $|\vec{r}_y \times \vec{r}_z| = \sqrt{4z^2+2} = \sqrt{2}\sqrt{2z^2+1}$, so

$$A(S) = \int_0^2\int_0^2 \sqrt{2}\sqrt{2z^2+1}\, dy\, dz$$

$$= 2\sqrt{2}\int_0^2 \sqrt{2z^2+1}\, dz$$

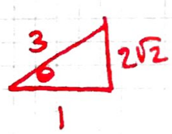
$\sqrt{2}\, z = \tan\theta \Rightarrow 2z^2+1 = \tan^2\theta + 1 = \sec^2\theta$

$\sqrt{2}\, dz = \sec^2\theta\, d\theta$

$$= 2\int_0^{\tan^{-1}(2\sqrt{2})} \sqrt{\sec^2\theta}\, \sec^2\theta\, d\theta = 2\int_0^{\tan^{-1}(2\sqrt{2})} \sec^3\theta\, d\theta$$

(parts) $$= 2\left[\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\log|\sec\theta + \tan\theta|\right]_0^{\tan^{-1}(2\sqrt{2})}$$

$$= \sec(\tan^{-1}(2\sqrt{2}))\cdot 2\sqrt{2} + \log|\sec(\tan^{-1}(2\sqrt{2})) + 2\sqrt{2}|$$



$$= 3\cdot 2\sqrt{2} + \log(3+2\sqrt{2})$$

$$= 6\sqrt{2} + \log(3+2\sqrt{2})$$

__Note__ Could also use #21 in table of integrals in back of the book.

16.7 Surface integrals

Just as we took line integrals of a function $f(x,y)$ over various curves, we introduce the notion of a surface integral for a function $f(x,y,z)$. Given a function of 3 variables f, the surface integral of f over S is

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA.$$

Notice the similarity of the equation

$$\int_C f(x,y,z)\, ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\, dt.$$

Also,

$$\iint_S 1\, dS = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA = A(S)$$

just as integrating the function $f(x,y,z) = 1$ over a curve C gives the length of C.

Example 4 Evaluate $\iint_S xyz\, dS$, where S is the cone given by $x = u\cos v$, $y = u\sin v$, $z = u$, $0 \le u \le 1$, $0 \le v \le \pi/2$.

Solution $\vec{r}(u,v) = \langle u\cos v, u\sin v, u\rangle \Rightarrow \vec{r}_u = \langle \cos v, \sin v, 1\rangle$, $\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$
So $\vec{r}_u \times \vec{r}_v = \langle -u\cos v, -u\sin v, u\rangle \Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2}\, u$
($\sqrt{u^2} = u$ since $u \ge 0$). Then

$$\begin{aligned}\iint_S xyz\, dS &= \int_0^1\int_0^{\pi/2} (u\cos v)(u\sin v)(u)\sqrt{2}\, u\, dv\, du \\ &= \int_0^1 \sqrt{2}\, u^4\, du \int_0^{\pi/2} \cos v \sin v\, dv \\ &= \left(\frac{\sqrt{2}}{5} u^5\Big|_0^1\right)\left(\frac{1}{2}\sin^2 v\Big|_0^{\pi/2}\right) \\ &= \sqrt{2}\cdot\frac{1}{5}\cdot\frac{1}{2} = \boxed{\frac{1}{10}\sqrt{2}}\end{aligned}$$

Example 5 Evaluate $\iint_S y^2\, dS$, where S is the part of the sphere $x^2+y^2+z^2=1$ that lies above the cone $z=\sqrt{x^2+y^2}$.

Solution The cone intersects the sphere at $x^2+y^2=\frac{1}{2}$, $z=\frac{1}{\sqrt{2}}$.

We parametrize with spherical coords: $\vec{r}(\varphi,\theta)=\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle$, $0\le\varphi\le\pi/4$, $0\le\theta\le 2\pi$. Now $\vec{r}_\varphi=\langle\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi\rangle$ and $\vec{r}_\theta=\langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0\rangle$, so that

$$\vec{r}_\varphi\times\vec{r}_\theta=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi\\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0\end{vmatrix}$$

$$=\sin^2\varphi\cos\theta\,\hat{\imath}+\sin^2\varphi\sin\theta\,\hat{\jmath}+\sin\varphi\cos\varphi\,\hat{k}$$

$$\Rightarrow |\vec{r}_\varphi\times\vec{r}_\theta|=\sqrt{\sin^4\varphi\cos^2\theta+\sin^4\varphi\sin^2\theta+\sin^2\varphi\cos^2\varphi}$$

$$=\sqrt{\sin^4\varphi+\sin^2\varphi\cos^2\varphi}$$

$$=\sqrt{\sin^2\varphi}=\sin\varphi \quad \text{since } 0\le\varphi\le\pi/4.$$

Now

$$\iint_S y^2\,dS=\int_0^{2\pi}\int_0^{\pi/4}(\sin\varphi\sin\theta)^2\sin\varphi\,d\varphi\,d\theta$$

$$=\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{\pi/4}\sin^3\varphi\,d\varphi$$

$$=\int_0^{2\pi}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta\int_0^{\pi/4}(1-\cos^2\varphi)\sin\varphi\,d\varphi$$

$$=\left[\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{3}\cos^3\varphi-\cos\varphi\right]_0^{\pi/4}$$

$$=\pi\left(\tfrac{\sqrt{2}}{12}-\tfrac{\sqrt{2}}{2}-\tfrac{1}{3}+1\right)$$

$$=\left(\tfrac{2}{3}-\tfrac{5\sqrt{2}}{12}\right)\pi.$$