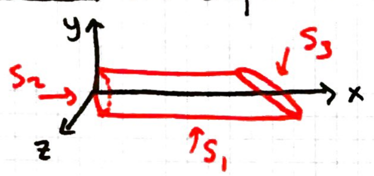
16.7 Surface integrals (cont'd)

Last time we learned how to compute $\iint_S f\,dS$, where S is a smooth surface. As with line integrals, we can relax this to S being piecewise-smooth, by computing the surface integral of each smooth piece and adding the result together (assuming S is a finite union of smooth surface that intersect only along their boundaries).

Example 1 Evaluate $\iint_S xz\,dS$, where S is the boundary of the region enclosed by the cylinder $y^2+z^2=9$, and the planes $x=0$ and $x+y=5$.

Solution. The surface S looks roughly like, where we have rotated the normal viewing angle.



$S_1: y^2+z^2=9$

$S_2: x=0$

$S_3: x+y=5.$

For S_1, we parametrize $\vec{r}(\theta,x)=\langle x, 3\cos\theta, 3\sin\theta\rangle$, $0\le\theta\le 2\pi$, $0\le x\le 5-y=5-3\cos\theta$

Now $\vec{r}_\theta=\langle 0,-3\cos\theta, 3\sin\theta\rangle$, $\vec{r}_x=\langle 1,0,0\rangle$, and
$\vec{r}_\theta\times\vec{r}_x=\langle 0, 3\cos\theta, 3\sin\theta\rangle \Rightarrow |\vec{r}_\theta\times\vec{r}_x|=3.$

Thus over S_1,

$$\begin{aligned}\iint_{S_1} xz\,dS &= \iint_D x(3\sin\theta)|\vec{r}_\theta\times\vec{r}_x|\,dA\\ &= \int_0^{2\pi}\int_0^{5-3\cos\theta} 9x\sin\theta\,dx\,d\theta\\ &= \int_0^{2\pi}\frac{9}{2}(5-3\cos\theta)^2\sin\theta\,d\theta\\ &= \frac{9}{2}\left[\frac{1}{9}(5-3\cos\theta)^3\right]_0^{2\pi} = 0\end{aligned}$$

Over S_2: $x=0$, so $\iint_{S_2} xz\,dS=\iint_S 0\,dS=0.$

Over S_3: $\vec{r}(y,z)=\langle 5-y, y, z\rangle \Rightarrow \vec{r}_y=\langle -1,1,0\rangle$, $\vec{r}_z=\langle 0,0,1\rangle$
$\Rightarrow |\vec{r}_y\times\vec{r}_z|=|\langle 1,1,0\rangle|=\sqrt{2}.$

$$\iint_{S_2} xz\,dS=\iint_{y^2+z^2\le 9}\sqrt{2}(5-y)z\,dA=\int_0^{2\pi}\int_0^3\sqrt{2}(5-r\cos\theta)\,r\sin\theta\,r\,dr\,d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\int_0^3(5r^2-r^3\cos\theta)\sin\theta\,dr\,d\theta=\sqrt{2}\int_0^{2\pi}\left(45-\frac{81}{4}\cos\theta\right)\sin\theta\,d\theta$$

$=0.$

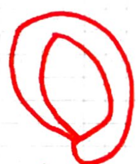
So $\iint_S xz\,dS=\iint_{S_1}xz\,dS+\iint_{S_2}xz\,dS+\iint_{S_3}xz\,dS=0+0+0=0.$

Oriented Surfaces

To define a surface integral of a vector field we need to give S an orientation. First, a nonexample. Not all surfaces S are orientable, ie, not all surfaces can be given an orientation. The classical nonexample is the Möbius strip.

To construct a Möbius strip, start with a strip of paper, make a half twist, then join the ends together. This surface can be obtained by the parametric equations

$$x = 2\cos\theta + r\cos\left(\tfrac{\theta}{2}\right)$$
$$y = 2\sin\theta + r\cos(\theta/2)\ , \quad 0 \le \theta \le 2\pi, \quad -\tfrac{1}{2} \le r \le \tfrac{1}{2}.$$
$$z = r\sin(\theta/2)$$



This is a bad picture. Look it up.

What's different about the Möbius strip is that it is "one-sided," in the sense that if you were to live on a Möbius strip and walk a straight path until you returned to your original location, you would be upside down.

What this means is that the unit normal vector doesn't vary continuously on the Möbius strip. In fact, it's not even well-defined since each point on the surface should have its unit normal vector pointing in two different directions.

From now on, we consider only orientable surfaces. If we have a surface S that has a tangent plane at every point (except its boundary), there are two normal vectors for each point, $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$.

If we can choose $\vec{n}$ at every point of S such that $\vec{n}$ varies continuously over S, then S is called an oriented surface; the choice of $\vec{n}$ gives S an orientation. For any orientable surface, there are two possible orientations.

If S is given by $z = g(x,y)$, then

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|} = \frac{\langle -\frac{\partial g}{\partial x}, -\partial g/\partial y, 1\rangle}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

gives the upward orientation of S (since the $\vec{k}$-component is positive).

If S is a smooth orientable surface given by $\vec{r}(u,v)$, then it is oriented by

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

and the opposite orientation is given by $-\vec{n}$.

A closed surface is a surface that is the boundary of a solid region E. By convention, the positive orientation is the one where the normal vectors point outward from E.

Positive orientation negative orientation

Example 2 The sphere $x^2+y^2+z^2=a^2$ is parametrized by

$$\vec{r}(\varphi,\theta) = \langle a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi \rangle,\quad 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi.$$

We've seen that $|\vec{r}_\varphi \times \vec{r}_\theta| = a^2\sin\varphi$, and $\vec{r}_\varphi \times \vec{r}_\theta = \langle a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\varphi \rangle$

Then the orientation induced by $\vec{r}(\varphi,\theta)$ is

$$\vec{n} = \frac{\vec{r}_\varphi \times \vec{r}_\theta}{|\vec{r}_\varphi \times \vec{r}_\theta|} = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle = \frac{1}{a}\vec{r}(\varphi,\theta).$$

Notice that the induced orientation is the positive one.

Surface integrals of vector fields

If $\vec{F}$ is a continuous vector field defined on an oriented surface S, with unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS.$$

This is also called the flux of $\vec{F}$ across S.

If S is given by $\vec{r}(u,v)$, then

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\, dS = \iint_D \left[\vec{F}(\vec{r}(u,v))\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\right]|\vec{r}_u\times\vec{r}_v|\, dA$$

Thus, $$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\, dA}$$

Notice This is similar to

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt.$$

Example 3 Find the flux of $\vec{F}$ across S for
$\vec{F}(x,y,z) = \langle z, y, z\rangle$, S the helicoid given by
$\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$, upward orientation

Solution
$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$ $\vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle \Rightarrow \vec{r}_u \times \vec{r}_v = \langle \sin v, -\cos v, u\rangle$
Also, $\vec{F}(r(u,v)) = \langle v, u\sin v, u\cos v\rangle$. So

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

$$= \int_0^1\int_0^\pi \langle v, u\sin v, u\cos v\rangle\cdot\langle \sin v, -\cos v, u\rangle\,dv\,du$$

Integration by parts.

$$= \int_0^1\int_0^\pi (\boxed{v\sin v} - u\cos v\sin v + u^2\cos v)\,dv\,du$$

$$= \int_0^1 \left[\sin v - v\cos v - \tfrac{1}{2}u\sin^2 v + u^2\sin v\right]_{v=0}^{v=\pi} du$$

$$= \int_0^1 \pi\,du = \pi.$$

If $\vec{F} = \langle P, Q, R\rangle$ and S is a graph of $z = g(x,y)$, then

$\vec{F}\cdot(\vec{r}_x \times \vec{r}_y) = \langle P, Q, R\rangle\cdot\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\rangle$, so

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA.$$

As before $\vec{r}(x,y)$ induces the (positive) upward orientation.

Example 4 Find the flux of $\vec{F}$ across S, where $\vec{F}(x,y,z) = \langle y, -x, 2z\rangle$ and S is the hemisphere $x^2+y^2+z^2 = 4$, $z \ge 0$, oriented downward.

Solution Here $g(x,y) = \sqrt{4-x^2-y^2}$, so by the above discussion,

$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_D \left(-y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} + 2z\right) dA,$$

where $D = \{(x,y) \mid x^2+y^2 \le 4\}$.

So,

$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_D \left[-y\cdot\tfrac{1}{2}(4-x^2-y^2)^{-\frac{1}{2}}(-2x) + x\cdot\tfrac{1}{2}(4-x^2-y^2)^{-\frac{1}{2}}(-2y) + 2z\right]dA$$

$$= -\iint_D \left[\frac{xy}{\sqrt{4-x^2-y^2}} - \frac{xy}{\sqrt{4-x^2-y^2}} + 2\sqrt{4-x^2-y^2}\right] dA$$

$$= -\iint_D 2\sqrt{4-x^2-y^2}\, dA$$

$$= -2\int_0^{2\pi}\int_0^2 \sqrt{4-r^2}\, r\, dr\, d\theta = \boxed{-\frac{32}{3}\pi.}$$

Example 5 Find the flux of $\vec{F}$ across S using the positive orientation where

$$\vec{F}(x,y,z) = \langle x, y, z^2\rangle$$

and S is the sphere of radius 2 centered at the origin.

Solution $\vec{r}(\varphi,\theta) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi\rangle$, $0\le\varphi\le\pi$, $0\le\theta\le 2\pi$.

From Example 2, $\vec{r}_\varphi\times\vec{r}_\theta = \langle 4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\theta\rangle$.

Also, $\vec{F}(\vec{r}(\varphi,\theta)) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 4\cos^2\varphi\rangle$

$$\vec{F}(\vec{r}(\varphi,\theta))\cdot(\vec{r}_\varphi\times\vec{r}_\theta) = 8\sin^3\varphi\cos^2\theta + 8\sin^3\varphi\sin^2\theta + 16\sin\varphi\cos^3\varphi$$
$$= 8\sin^3\varphi + 16\sin\varphi\cos^3\varphi$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi}(8\sin^3\varphi + 16\sin\varphi\cos^3\varphi)\,d\varphi\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} 8\sin\varphi(1-\cos^2\varphi + 2\cos^3\varphi)\,d\varphi$$

$$= 2\pi\cdot 8\int_{-1}^{1}(1-u^2+2u^3)\,du$$

$$= 16\pi\left[u - \tfrac{1}{3}u^3 + \tfrac{1}{2}u^4\right]_{-1}^{1}$$

$$= 16\pi\left[1-\tfrac{1}{3}+\tfrac{1}{2} - \left(-1+\tfrac{1}{3}+\tfrac{1}{2}\right)\right]$$

$$= \frac{64\pi}{3}$$