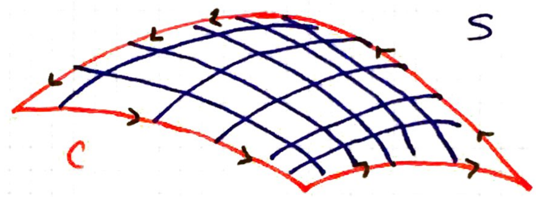
16.8 Stokes' Theorem

In this section S will be an oriented piecewise-smooth surface, bounded by a simple closed piecewise smooth curve $C = \partial S$.

The orientation of S induces the positive orientation of ∂S. This means that if you walk in the positive direction along ∂S with your head pointing in the direction of $\vec{n}$, (the unit normal vector orienting S) then the surface will always be on your left.



<u>Stokes' Theorem</u> Let S and ∂S be as above. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives. on an open region in $\mathbb{R}^3$ Containing S. Then

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

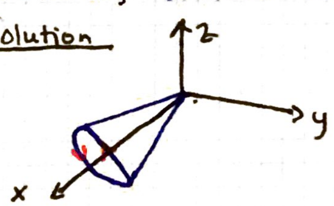
<u>Remark</u> If S is flat and lies in the xy-plane with upward orientation, then its unit normal is $\hat{k}$, and Stokes' Thm says

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S (\text{curl}\,\vec{F})\cdot \hat{k}\, dA,$$

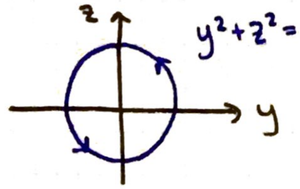
which is precisely the vector form of Green's Theorem. So Green's Thm is really a special case of Stokes'.

<u>Example 1</u> Use Stokes' Thm to evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z) = \tan^{-1}(x^2yz^2)\hat{\imath} + x^2y\hat{\jmath} + x^2z^2\hat{k}$, S is the cone $x = \sqrt{y^2+z^2}$, $0 \le x \le 2$, oriented in the direction of the positive x-axis.

<u>Solution</u>



The normal vectors point into the cone. The boundary curve of S is $2 = \sqrt{z^2+y^2} \iff y^2+z^2=4$, $x=2$. This curve is oriented counter-clockwise when viewed from the positive x-axis.



$y^2+z^2=4,\ x=2$

We can parametrize ∂S by

$\vec{r}(t) = \langle 2, 2\cos t, 2\sin t\rangle,\quad 0 \le t \le 2\pi.$

$d\vec{r} = \langle 0, -2\sin t, 2\cos t\rangle\, dt.$

So, $\vec{F}(\vec{r}(t))\cdot d\vec{r} = \langle \tan^{-1}(32\cos t\sin^2 t), 8\cos t, 16\sin^2 t\rangle \cdot \langle 0, -2\sin t, 2\cos t\rangle dt$

$= 0 - 16\cos t\sin t\, dt + 32\sin^2 t\cos t\, dt$

Thus, by Stokes' Thm, $\iint_S \vec{F}\cdot d\vec{S}$ is given by

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (-16\cos t\sin t + 32\sin^2 t\cos t)\, dt$$

$$= \int_0^{2\pi} (-16\sin t + 32\sin^2 t)\cos t\, dt \qquad \begin{aligned} u &= \sin t \\ du &= \cos t\, dt\end{aligned}$$

$$= \int_0^0 (-16u + 32u^2)\, du$$

$$= 0.$$

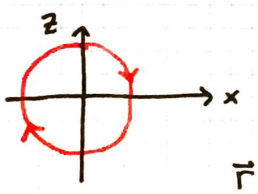
<u>Notice</u> When we use Stokes' Thm in this direction, we only need to know the values of $\vec{F}$ on ∂S. This means that if $\partial S_1 = C = \partial S_2$, and S_1 and S_2 satisfy the hypotheses of Stokes Thm,

$$\boxed{\iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \iint_{S_2} \operatorname{curl}\vec{F}\cdot d\vec{S}}$$

We can use this fact if it's hard to integrate over one surface but easy to integrate over another if they have the same boundary curve.

<u>Example 2</u> Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle e^{xy}, e^{xz}, x^2 z\rangle$, S is the half of the ellipsoid $4x^2+y^2+4z^2=4$ that lies to the right of the xz-plane oriented in the direction of the positive y-axis.

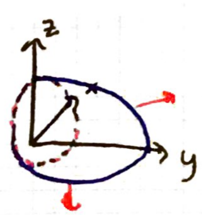
<u>Solution</u> The boundary curve of S is $x^2+z^2=1,\ y=0$.



When viewed from the xz-plane, the curve is oriented <u>clockwise</u>. Thus ∂S is parametrized by

$\vec{r}(t) = \langle \cos(-t), 0, \sin(-t)\rangle,\quad 0\le t\le 2\pi$

$= \langle \cos t, 0, -\sin t\rangle,\quad 0 \le t \le 2\pi.$



Now $d\vec{r} = \langle -\sin t, 0, -\cos t \rangle \, dt$

$$\vec{F}(\vec{r}(t)) = \langle 1, e^{-\cos t \sin t}, -\cos^2 t \sin t \rangle$$

$$\vec{F}(\vec{r}(t)) \cdot d\vec{r} = (-\sin t + \cos^3 t \sin t)\, dt$$
$$= (1 - \cos^3 t)(-\sin t)\, dt.$$

So by Stokes' Thm,

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$
$$= \int_0^{2\pi} (1 - \cos^3 t)(-\sin t)\, dt \qquad \begin{matrix} u = \cos t \\ du = -\sin t \, dt \end{matrix}$$
$$= \int_1^1 (1 - u^3)\, du$$
$$= 0.$$

There are instances (much like Green's Thm), where it is useful to use Stokes' Thm in the opposite direction.

Example 3 Use Stokes' Thm to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is oriented counterclockwise when viewed from above.

$$\vec{F}(x,y,z) = \langle 2y, xz, x+y \rangle.$$

C is the curve of intersection of the plane $z = y+2$ and the cylinder $x^2 + y^2 = 1$.

Solution

C

The curve C is an ellipse in the plane $z = y+2$. We compute

$$\operatorname{curl} \vec{F} = \nabla \times \langle 2y, xz, x+y \rangle = \langle 1-x, -1, z-2 \rangle.$$

Then we take S to be given by $g(x,y) = y+2$ with $x^2+y^2 \le 1$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_{x^2+y^2 \le 1} \left(-(1-x)(0) + (1)(1) + y + 2 - 2 \right)$$

$$= \iint_{x^2+y^2 \le 1} (y+1)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (r \sin\theta + 1)\, r \, dr \, d\theta = \pi.$$

<u>Example 4</u> Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = \langle x^2 y, \frac{1}{3}x^3, xy \rangle$, and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

<u>Solution</u> We make S the part of the surface $z = y^2 - x^2$ that lies above the disk $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$. Now $\text{curl}\,\vec{F} = \langle x, -y, 0 \rangle$, and we use $g(x,y) = y^2 - x^2$. By Stokes' Thm,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

$$= \iint_D \left(-x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} + 0\right) dA$$

$$= \iint_D (2x^2 + 2y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 2r^3\, dr\, d\theta = \pi.$$