16.9 The Divergence Theorem

The Divergence Theorem Let S be a closed region with positive (outward) orientation, and let E be the solid region enclosed by S. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region containing E. Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV.$$

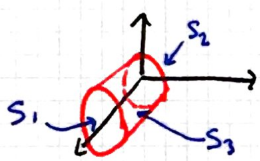
In this situation, the flux of $\vec{F}$ across the boundary surface of E is equal to the divergence of $\vec{F}$ over E.

Example 1 Verify that the Divergence Theorem is true for

$$\vec{F}(x,y,z) = \langle x^2, -y, z\rangle,$$

E is the solid cylinder $y^2+z^2 \le 9,\ 0\le x\le 2.$

Solution We first compute $\iint_S \vec{F}\cdot d\vec{S}$ directly.



Let S_1 be the part of the cylinder in the $x=2$ plane.
S_2 the part of the cylinder in the $x=0$ plane.
S_3 the lateral surface $y^2+z^2=9$.

S_1 is the disk $y^2+z^2\le 9$, $x=2$ with unit normal vector $\vec{n}=\langle 1,0,0\rangle$. On S_1, $\vec{F}=\langle 4,-y,z\rangle$. Thus

$$\iint_{S_1}\vec{F}\cdot d\vec{S} = \iint_{S_1}\vec{F}\cdot\vec{n}\,dS = \iint_{S_1} 4\,dS = 4\cdot\pi(3)^2 = 36\pi$$

S_2 is the disk $y^2+z^2\le 9$, $x=0$ with unit normal $\vec{n}=\langle -1,0,0\rangle$ and here $\vec{F}=\langle 0,-y,z\rangle$, so

$$\iint_{S_2}\vec{F}\cdot d\vec{S} = \iint_{S_2}\vec{F}\cdot\vec{n}\,dS = \iint_{S_2} 0\,dS = 0.$$

We parametrize S_3 by $\vec{r}(x,\theta)=\langle x, 3\cos\theta, 3\sin\theta\rangle$. Then $\vec{r}_x\times\vec{r}_\theta = \langle 1,0,0\rangle\times\langle 0,-3\sin\theta,3\cos\theta\rangle = \langle 0,-3\cos\theta,-3\sin\theta\rangle$. Note that to give S_3 the positive orientation, we need to use $\vec{n}=-(\vec{r}_x\times\vec{r}_\theta)$. Also, $\vec{F}(\vec{r}(x,\theta)) = \langle x^2, -3\cos\theta, 3\sin\theta\rangle$, so

$$\iint_{S_3}\vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot -(\vec{r}_x\times\vec{r}_\theta)\,dA = \int_0^2\int_0^{2\pi}(0-9\cos^2\theta+9\sin^2\theta)\,d\theta\,dx$$
$$= -9\int_0^2 dx\int_0^{2\pi}\cos 2\theta\,d\theta$$
$$=0.$$

Thus $\iint_S \vec{F} \cdot d\vec{S} = 36\pi + 0 + 0 = 36\pi.$

On the other hand, $\operatorname{div} \vec{F} = 2x - 1 + 1 = 2x$, so

$$\iiint_E \operatorname{div} \vec{F} \, dV = \iint_{y^2+z^2 \le 9} \left[\int_0^2 2x \, dx \right] dA$$

$$= \iint_{y^2+z^2 \le 9} 4 \, dA$$

$$= 4 \cdot \pi (3)^2 = 36\pi.$$

Remark The above example shows that the divergence theorem can be much more efficient for computing the flux of $\vec{F}$ across S when S is a closed surface.

Example 2 Compute the integral $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle x^3 + y^3, y^3 + z^3, x^3 + z^3 \rangle$, S is the sphere with radius 2, center the origin.

Solution It's really simple to compute triple integrals in spherical coords, so it's helpful to use the Divergence Thm here. Now,

$\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 3z^2$, so the divergence Thm says

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E 3(x^2+y^2+z^2) \, dV$$

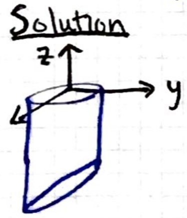
$$= \int_0^{\pi} \int_0^{2\pi} \int_0^2 3\rho^2 \cdot \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi.$$

$$= 3 \int_0^{\pi} \sin\varphi \, d\varphi \int_0^{2\pi} d\theta \int_0^2 \rho^4 \, d\rho$$

$$= \frac{384}{5} \pi.$$

Example 3 Evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle x^2 + y^2, xy + 2yz, xy - z^2 \rangle$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = x - 2$ and $z = 0$.

Solution



Since $|x| \le 2$, we have $x - 2 \le z \le 0$. We'll use the Divergence Thm.

$$\operatorname{div} \vec{F} = 2x + x + 2z - 2z = 3x.$$

Thus, by the Div. Thm,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV.$$ We convert to cylindrical coords.

$$E = \{ (r,\theta,z) \mid r\cos\theta - 2 \le z \le 0, \ 0 \le r \le 2, \ 0 \le \theta \le 2\pi \}$$

So the integral above becomes

$$\iiint_E 3x \, dV = \int_0^{2\pi} \int_0^2 \int_{r\cos\theta - 2}^{0} (3r\cos\theta) \, r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left(-3r^3 \cos^2\theta + 6r^2 \cos\theta \right) dr \, d\theta$$

$$= \int_0^{2\pi} \left(-12\cos^2\theta + 16\cos\theta \right) d\theta$$

$$= -12\pi.$$

Summary

To compute $\iint_S \vec{F} \cdot d\vec{S}$, we can do this by

- Directly: $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS$ ($\vec{n}$ = unit normal)

  Parametrizing S by $\vec{r}(u,v)$ with $(u,v) \in D$, this becomes

  $$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dA.$$

- If S is closed, we can use the divergence thm:

  $$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV,$$ E enclosed by S.

- If S is bounded by a simple closed curve C, then

  $$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

To compute $\int_C \vec{F} \cdot d\vec{r}$, we can do this by

- If $\vec{F}$ is conservative, find f such that $\vec{F} = \nabla f$, then evaluate at terminal and initial points (Fundamental Thm).
- If C is not closed,

  $$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

- If C is closed, then we can use Green's ($\mathbb{R}^2$) or Stokes' ($\mathbb{R}^3$)

  $$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

  $$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}.$$

To compute

- $\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$
- $\int_C f(x,y)\,dx = \int_a^b f(\vec{r}(t))\,x'(t)\,dt$
- $\int_C f(x,y)\,dy = \int_a^b f(\vec{r}(t))\,y'(t)\,dt$

In the above $\vec{r}(t) = \langle x(t), y(t) \rangle$ parametrizes C for $a \le t \le b$.