## 13.1 Vector functions and space curves

A vector-valued function is a function whose output is a vector. We are usually interested in vector-functions from $\mathbb{R} \to \mathbb{R}^3$.

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$f, g, h$ called component functions. Usually think of $t$ as time.

The domain of $\vec{r}(t)$ is the intersection of the domains of $f, g, h$.

Example 1 Find the domain of $\vec{r}(t) = \langle \cos t, \log t, \frac{1}{\sqrt{t-1}} \rangle$

Solution domain of
$\cos t$ : $\mathbb{R}$
$\log t$ : $t > 0$
$1/\sqrt{t-1}$ : $t > 1$

$\Rightarrow$ domain of $\vec{r}(t)$ : $(t, \infty)$.

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

provided the limits of the component functions exist.

Naturally, a vector function $\vec{r}(t)$ is continuous at $a$ if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

Remark A vector function is continuous iff its component functions are.

Space Curves If $C$ is a curve whose points are $(x, y, z)$ such that

$$x = f(t), \quad y = g(t), \quad z = h(t) \quad (*)$$

for all $t$ in some interval $I$, then $C$ is called a space curve. The equations $(*)$ are parametric eqns in the parameter $t$.

Example 2 Sketch the given curve.

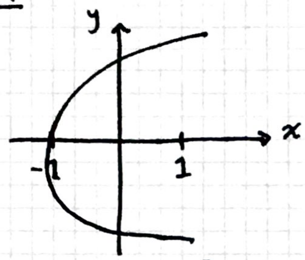
(a) $\vec{r}(t) = \langle t^2 - 1, t \rangle$

(b) $\vec{r}(t) = \langle 2\cos t, 2\sin t, 1 \rangle$

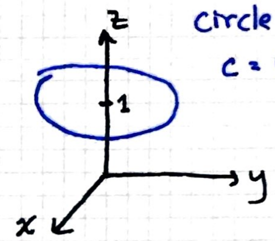
(c) $\vec{r}(t) = \langle \cos t, t, \sin t \rangle$

Solution

(a)

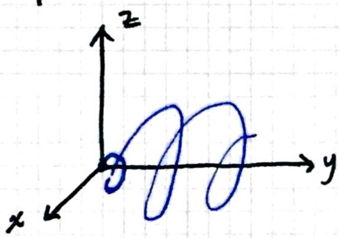



(b)




circle $r = 2$
$c = (0,0,1)$

(c)




Circular helix down y-axis

Example 3 Find vector eqn joining $P(a,b,c)$ to $Q(x,y,z)$.

Solution From 12.5, $\vec{r}(t) = (1-t)\vec{r}_0 + t\,\vec{r}_1$, $0 \le t \le 1$.

$$\vec{r}(t) = (1-t)\langle a,b,c \rangle + t\langle x,y,z \rangle$$

Example 4 Find a vector function that represents the curve of intersection of $z = x^2 - y^2$ and $x^2 + y^2 = 1$.

Solution Projecting onto the xy-plane: $x^2 + y^2 = 1$.
So $x = \cos t$, $y = \sin t$. Then $z = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 = \cos 2t$
So the curve of intersection is

$$\vec{r}(t) = \langle \cos t, \sin t, \cos 2t \rangle$$

Example 5 At what points does the helix $\vec{r}(t) = \langle \sin t, \cos t, t \rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

Solution $\vec{r}(t)$ has $x = \sin t$, $y = \cos t$, $z = t$ for all $t$.
Substituting into the sphere, we get

$$\sin^2 t + \cos^2 t + t^2 = 5$$
$$1 + t^2 = 5$$
$$\boxed{t = \pm 2}$$

So, they intersect at the points $(\sin(2), \cos(2), 2)$ and $(\sin(-2), \cos(-2), -2)$

## 13.2 Derivatives and integrals of Vector functions

Definition same as Calc 1:

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

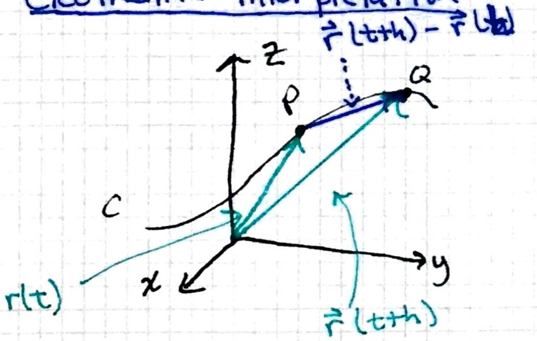
(if the limit exists).

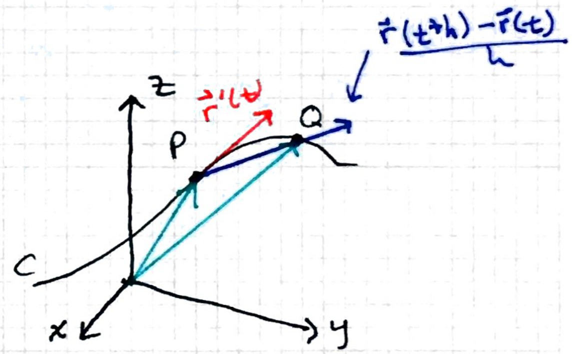
To compute, we use

**Theorem** If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $f, g, h$ diff'ble functions then $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

proof Follows from the fact that the limit of $\vec{r}(t)$ is the limit of its component functions and def of $d/dt$.

Geometric interpretation




If $\vec{r}\,'(t)$ exists and $\vec{r}\,'(t) \neq \vec{0}$, then $\vec{r}\,'(t)$ called the tangent vector. Often want to consider unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

**Example 6**

(a) Find the derivative of $\vec{r}(t) = \langle \cos t + 1, \sin t - 1 \rangle$, sketch $\vec{r}(-\pi/3)$, $\vec{r}\,'(-\pi/3)$.

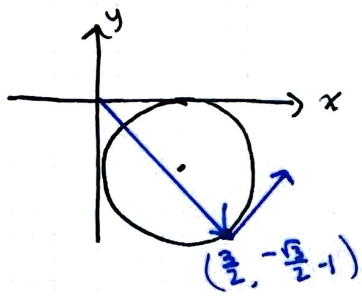
(b) Find $\vec{T}$ for $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

**Solution**

(a) $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$\vec{r}(-\frac{\pi}{3}) = \langle \frac{3}{2}, \frac{-\sqrt{3}}{2} - 1 \rangle$

$\vec{r}\,'(-\frac{\pi}{3}) = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$




(b) $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle \Rightarrow \vec{r}\,'(t) = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$\vec{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle.$

**Differentiation Rules** $\vec{u}, \vec{v}$ diff'ble vec fcns, $c \in \mathbb{R}$, $f$ $\mathbb{R}$-val fcn.

1) $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}\,'(t) + \vec{v}\,'(t)$

2) $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}\,'(t)$

3) $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$

4) $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$

5) $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$

6) $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}\,'(f(t))$ Chain Rule

*Proof* Can be done using Thm about d/dt. For example, to Prove 4). Let $\vec{u}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$, $\vec{v}(t) = \langle g_1(t), g_2(t), g_3(t) \rangle$
Then $\vec{u} \cdot \vec{v} = f_1 g_1 + f_2 g_2 + f_3 g_3$. One-var prod rule gives

$$\begin{aligned} \frac{d}{dt}(u \cdot v) &= \frac{d}{dt}(f_1 g_1 + f_2 g_2 + f_3 g_3) \\ &= f_1' g_1 + f_1 g_1' + f_2' g_2 + f_2 g_2' + f_3' g_3 + f_3 g_3' \\ &= (f_1' g_1 + f_2' g_2 + f_3' g_3) + (f_1 g_1' + f_2 g_2' + f_3 g_3') \\ &= u' \cdot v + u \cdot v'. \checkmark \end{aligned}$$

<u>Integrals</u> work same way, follows from limit def.

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

~~Example 7 Evaluate~~

$\cancel{\int_0^1}$

<u>FTC</u> $\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$, where $\vec{R}'(t) = \vec{r}(t)$.

<u>Example 7</u> Evaluate $\int_0^1 \vec{r}(t)\,dt$, where

$$\vec{r}(t) = \left\langle \frac{1}{t+1},\ \frac{1}{t^2+1},\ \frac{t}{t^2+1} \right\rangle$$

$$\int_0^1 \vec{r}(t)\,dt = \left\langle \log(t+1)\Big|_0^1,\ \tan^{-1}(t)\Big|_0^1,\ \frac{1}{2}\log u\ \Big|_1^2 \right\rangle$$

$$= \left\langle \log 2,\ \frac{\pi}{4},\ \frac{1}{2}\log 2 \right\rangle.$$