13.3 Arc length and Curvature.

From Calc II, know arc length of plane curve given by $x = f(t)$, $y=g(t)$ from a to b is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Similar formula holds for space curve. Notice also

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_a^b |\vec{r}\,'(t)|\, dt$$

Arc length is integral of speed.

Example 1 Find arclength of

(a) $\vec{r}(t) = \langle \cos t, \sin t, \log(\cos t) \rangle \quad 0 \le t \le \frac{\pi}{4}$

(b) $\vec{r}(t) = \langle t^2, 9t, 4t^{3/2} \rangle \quad 1 \le t \le 4$

Solution (a) $\vec{r}\,'(t) = \langle -\sin t, \cos t, \frac{-\sin t}{\cos t} \rangle$

$$\vec{r}\,'(t) = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} = \sqrt{1+\tan^2 t}$$

$$= |\sec t|$$

$$= \sec t \quad \text{for } 0 \le t \le \frac{\pi}{4}$$

$$L = \int_0^{\frac{\pi}{4}} \sec t\, dt = \log(\sec t + \tan t)\Big|_0^{\pi/4}$$

$$= \log(\sqrt{2} + 1) - \log 1 = \log(\sqrt{2}+1)$$

(b) $\vec{r}\,'(t) = \langle 2t, 9, 6t^{1/2} \rangle$

$$\vec{r}\,'(t) = \sqrt{4t^2 + 81 + 36t} = \sqrt{(2t+9)^2} = |2t+9|$$

$$= 2t+9 \quad \text{for } 1 \le t \le 4$$

$$\Rightarrow L = \int_1^4 2t+9\, dt = t^2 + 9t\Big|_1^4 = 42$$

There are different parametrizations for the same curve. For example:

$$\vec{r}_1(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le \pi$$

$$\vec{r}_2(t) = \langle t, \sqrt{1-t^2} \rangle \quad |t| \le 1$$

parametrizes the upper half circle

Can be shown: arc length formula independent of parametrization.

Arc length function

C: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $\vec{r}\,'$ continuous, traverse C exactly once. for $a \le t \le b$. The arc length function of C is

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du$$

By FTC: $\frac{d}{dt} s(t) = |\vec{r}\,'(t)|$.
Often want to parametrize by arc length.
Idea: Given $\vec{r}(t)$ and a.l. function $s(t)$, solve for t as function of s to get $t = t(s)$ and replace $\vec{r} = \vec{r}(t(s))$.

Example 2. Reparametrize $\vec{r}(t) = \langle e^t \sin t, e^t \cos t, \sqrt{2} e^t \rangle$ starting from $t=0$ $t\nearrow$.

Solution $\vec{r}\,'(t) = \langle e^t(\sin t + \cos t), e^t(\cos t - \sin t), \sqrt{2} e^t \rangle$

$$\Rightarrow |\vec{r}\,'(t)| = [e^{2t}(\sin t + \cos t)^2 + e^{2t}(\cos t - \sin t)^2 + 2e^{2t}]^{\frac{1}{2}}$$

$$= (4e^{2t})^{1/2} = 2e^{t}$$

$$s(t) = \int_0^t 2e^u\, du = 2(e^t - 1)$$

Solving for t: $t = \log(\frac{1}{2}s + 1)$

$$\vec{r}(t(s)) = \langle (\tfrac{1}{2}s+1)\sin(\log(\tfrac{1}{2}s+1)), (\tfrac{1}{2}s+1)\cos(\log(\tfrac{1}{2}s+1)), \sqrt{2}(\tfrac{1}{2}s+1) \rangle$$

Curvature

A parametrization $\vec{r}(t)$ is called smooth on an interval I if $\vec{r}\,'$ is continuous and $\vec{r}\,'(t) \neq 0$ on I. A curve C is smooth if it has a smooth parametrization.

Def The curvature of a curve is

$$\kappa = \left| \frac{d\vec{T}}{ds} \right|$$ where $\vec{T}$ is unit tangent vector.

Alternatively, by the chain rule

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt} \Rightarrow \kappa = \left|\frac{d\vec{T}}{ds}\right| = \left| \frac{d\vec{T}/dt}{ds/dt} \right|$$

Using $\frac{ds}{dt} = \vec{r}\,'(t)$, $\kappa(t) = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$

Normal and Binormal vectors

Note that since $|\vec{T}(t)| = 1$, $\vec{T}'(t)\cdot\vec{T}(t) = 0$ (why?), so $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$. $\vec{T}'(t)$ usually not a unit vector.

If $\kappa \neq 0$, we define unit normal vector

$$\vec{N}(t) = \frac{T'(t)}{|T'(t)|}$$

The Binormal vector $\vec{B}(t) = \vec{T}(t)\times\vec{N}(t)$. Unit vector perpendicular to $\vec{T}(t)$ and $\vec{N}(t)$.

Example 4 (i) Compute $\vec{T}(t)$, $\vec{N}(t)$, (ii) Find κ.

(a) $\vec{r}(t) = \langle t, \frac{1}{2}t^2, t^2\rangle$

(b) $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$

~~(c) $\vec{r}(t) = \langle t, t^2, t^3\rangle$~~

Solution

(a) $\vec{r}'(t) = \langle 1, t, 2t\rangle$, $\quad \vec{T}(t) = \frac{1}{(1+5t^2)^{1/2}}\langle 1, t, 2t\rangle$

$|\vec{r}'(t)| = \sqrt{1+t^2+4t^2}$

$\cancel{\vec{N}(t)}\quad \vec{T}'(t) = \frac{-5t}{(1+5t^2)^{3/2}}\langle 1, t, 2t\rangle + \frac{1}{(1+5t^2)^{1/2}}\langle 0, 1, 2\rangle$

$= \frac{1}{(1+5t^2)^{3/2}}\langle -5t, 1, 2\rangle$

$$|\vec{T}'(t)| = \frac{1}{(1+5t^2)^{3/2}}\sqrt{25t^2+1+4} = \frac{\sqrt{5}\sqrt{1+5t^2}}{(1+5t^2)^{3/2}} = \frac{\sqrt{5}}{1+5t^2}$$

$$\Rightarrow \vec{N}(t) = \frac{1}{\sqrt{5+25t^2}}\langle -5t, 1, 2\rangle$$

(ii) $\kappa = \frac{|T'(t)|}{|r'(t)|} = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}$

(b) $\vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle \Rightarrow |\vec{r}'(t)| = \sqrt{2}$

$\vec{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle$

$\vec{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0\rangle$

$|\vec{T}'(t)| = \frac{1}{\sqrt{2}}\sqrt{\cos^2 t+\sin^2 t} = \frac{1}{\sqrt{2}} \Rightarrow N(t) = \langle -\sin t, \cos t, 0\rangle$

(ii) $\kappa = 1$.

Remark Sometimes it's easier to compute κ using

$$\kappa = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

Like in Example 4(a): $\vec{r}\,'(t) = \langle 1, t, 2t \rangle$, $\vec{r}\,''(t) = \langle 0, 1, 2 \rangle$

so $r' \times r'' = \langle 0, -2, 1 \rangle$. and

$$\kappa = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}$$

But in the homework you get forced to use $\kappa = \frac{|T'(t)|}{|r'(t)|}$