13.4 Velocity and acceleration

Same as calc 1:

$\vec{v}(t) = \vec{r}\,'(t)$ and $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

Also, speed $= |\vec{v}(t)|$. To find average velocity on $[a,b]$,

$$\vec{v}_{ave} = \frac{\vec{r}(b) - \vec{r}(a)}{b-a}.$$

Example 1 Find velocity and acceleration vectors, speed

$\vec{r}(t) = \langle t, 2\cos t, \sin t \rangle$

Solution $\vec{v}(t) = \langle 1, -2\sin t, \cos t \rangle$, $\vec{a}(t) = \langle 0, -2\cos t, -\sin t \rangle$

$|\vec{v}(t)| = \sqrt{1 + 4\sin^2 t + \cos^2 t} = \sqrt{2 + 3\sin^2 t}$

Example 2 Show that if a particle moves with constant speed, then velocity and acceleration vectors are orthogonal.

Solution Say $|\vec{v}(t)| = c$. Then $|\vec{v}(t)|^2 = \vec{v}(t)\cdot\vec{v}(t) = c^2$.
Differentiating both sides,

$$\vec{v}\,'(t)\cdot\vec{v}(t) + \vec{v}(t)\cdot\vec{v}\,'(t) = 2\vec{v}(t)\cdot\vec{v}\,'(t) = 0$$

But $\vec{v}\,'(t) = a(t) \Rightarrow 2\vec{v}(t)\cdot a(t) = 0 \Leftrightarrow \vec{v}(t)\cdot\vec{a}(t) = 0$.

Example 3 Find the position vector given

$$\vec{a}(t) = \langle 2t, \sin t, \cos 2t \rangle, \quad \vec{v}(0) = \langle 1, \overset{\hat{i}}{0}, 0 \rangle, \quad \vec{r}(0) = \langle 0, \overset{\hat{j}}{1}, 0 \rangle.$$

Solution $\vec{v}(t) = \int \vec{a}(t)\,dt + \vec{C}$, some constant vector $\vec{C}$.

$\vec{v}(t) = \langle t^2, -\cos t, \frac{1}{2}\sin 2t \rangle + \vec{C}$. Now $\langle 1, 0, 0 \rangle = \vec{v}(0) = \langle 0, -1, 0 \rangle + \vec{C}$
$\Rightarrow \vec{C} = \langle 1, 1, 0 \rangle. \Rightarrow \vec{v}(t) = \langle t^2 + 1, 1 - \cos t, \frac{1}{2}\sin 2t \rangle$. So,

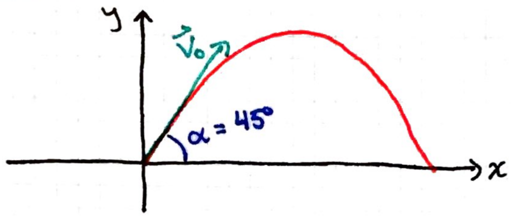
$\vec{r}(t) = \langle \frac{1}{3}t^3 + t, t - \sin t, -\frac{1}{4}\cos 2t \rangle + \vec{D}$. Now,

$\langle 0, 1, 0 \rangle = \vec{r}(0) = \langle 0, 0, -\frac{1}{4} \rangle + \vec{D} \Rightarrow \vec{D} = \langle 0, 1, \frac{1}{4} \rangle$, and

$\vec{r}(t) = \langle \frac{1}{3}t^3 + t, 1 + t - \sin t, \frac{1}{4} - \frac{1}{4}\cos 2t \rangle$.

Example 4 A ball is thrown at an angle of 45° to the ground. If the ball lands 90 m away, what was the initial speed of the ball?

Solution




Let $\vec{v}_0$ be the initial velocity. The force acting on the ball is gravity, so

$$\vec{F} = m\vec{a} = -mg\hat{j},$$

where $g = |\vec{a}| = 9.8\ m/s^2$. Therefore $\vec{a} = -g\hat{j}$. Now $\vec{v}'(t) = \vec{a}(t) \Rightarrow$

$$\vec{v}(t) = -gt\hat{j} + \vec{C}, \quad \text{where } \vec{C} = \vec{v}(0) = \vec{v}_0.$$

So,
$$\vec{r}'(t) = \vec{v}(t) = -gt\hat{j} + \vec{v}_0$$
$$\Rightarrow \vec{r}(t) = -\tfrac{1}{2}t^2 g\hat{j} + t\vec{v}_0 \quad (*)$$

$(+\vec{D}$, but $\vec{D} = \vec{r}(0) = \vec{0})$. The initial speed is $|\vec{v}_0|$, so

$$\vec{v}_0 = \langle |\vec{v}_0|\cos(45°), |\vec{v}_0|\sin(45°)\rangle$$
$$= \langle |\vec{v}_0|/\sqrt{2}, |\vec{v}_0|/\sqrt{2}\rangle.$$

Now putting this into $(*)$, $\vec{r}(t) = \langle t|\vec{v}_0|/\sqrt{2},\ t|\vec{v}_0|/\sqrt{2} - \tfrac{1}{2}gt^2\rangle$. The ball lands when $y = t|\vec{v}_0|/\sqrt{2} - \tfrac{1}{2}gt^2 = 0$ and $t > 0$. That is,

$t = \frac{2|\vec{v}_0|}{\sqrt{2}g} = \frac{\sqrt{2}|\vec{v}_0|}{g}$. Landing 90 m away $\Rightarrow 90 = x = \frac{|\vec{v}_0|}{\sqrt{2}} \cdot \frac{\sqrt{2}|\vec{v}_0|}{g}$

$\Rightarrow |\vec{v}_0|^2 = 90g. \Rightarrow |\vec{v}_0| = \sqrt{90g}$.