14.2 Limits and continuity.

Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then the <u>limit</u> of $f(x,y)$ as (x,y) approaches (a,b) is L, and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then

$$|f(x,y) - L| < \varepsilon.$$

Also may write $f(x,y) \to L$ as $(x,y) \to (a,b)$.

Remark This says if we get arbitrarily close to the point (a,b) (inside a disc of radius δ, center (a,b)), then we will be arbitrarily close to the value L.

Recall that in one variable, for a limit to exist, both the left- and right-hand limits must exist and be equal. With two variables, there is no "left" and "right", but we need the limit along <u>any</u> path to (a,b) to be the same.

In other words, if $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along C_2, $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does <u>not</u> exist.

Example 1 $f(x,y) = \dfrac{xy}{x^2+y^2}$, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

Solution Along the path $x=0$,

$$f(0,y) = \frac{0}{y^2} = 0, \quad \text{but along } y=x,$$

$$f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2}$$

So $f(x,y) \to 0$ along $x=0$, but $f(x,y) \to \frac{1}{2}$ along $y=x$.

So the limit does not exist.

A function f of two variables is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say f is continuous on D if f is continuous at every point (a,b) in D.

Strategy for computing limits

To compute $\lim_{(x,y)\to(a,b)} f(x,y) = L$,

1) If f is continuous at (a,b), then $L = f(a,b)$.

2) If f is not continuous at (a,b)
   (a) Try factoring / simplifying algebraically
   (b) If there's a $\sqrt{\ }$, try multiplying by the conjugate
   (c) If you suspect the limit DNE, try various paths
      - $x=0$
      - $y=mx$ (for fixed m)
      - $y=mx^2$ (for fixed m)

If these are insufficient, then you've probably made a mistake (in the context of this class only).

Example 2 Compute the limit, if it exists.

(a) $\lim_{(x,y)\to(0,1)} \sin\left(x e^{\sqrt{y+1}}\right)$

(b) $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$

(c) $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

(d) $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

Solution (a) $\sqrt{y+1}$ is continuous at 0, exponential and sine functions are always continuous, so the composition is continuous, so the limit is

$$\sin(0\cdot e^{\sqrt{2}}) = 0.$$

(b) Try $x=0$: get $0/y^4 = 0$.
Try $y=mx$: get

$$\frac{x(mx)^2}{x^2+(mx)^4} = \frac{m^2x^3}{x^2+m^4x^4} = \frac{m^2x^3}{x^2(1+m^4x^2)} = \frac{m^2x}{1+m^4x^2} \xrightarrow[x\to 0]{} 0.$$

But if we try $x=y^2$,

$$\frac{y^2y^2}{(y^2)^2+y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

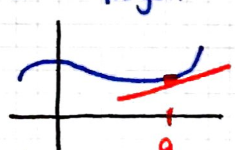
So the limit does not exist.

(c) Alon $x=0$, get $-y^2/y^2 = -1$, but along $y=0$, get $x^2/x^2 = 1$. So limit DNE.

(d)
$$\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} = \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{(x^2+y^2+1)-1}$$
$$= \sqrt{x^2+y^2+1}+1 \longrightarrow 2 \text{ as } (x,y)\to(0,0).$$

14.3 Partial derivatives

In one variable, $y=f(x)$, $\frac{dy}{dx} = f'(x)$ gives the rate of change or the slope of the tangent line.



$y=f(x)$

But a function of two variables $z=f(x,y)$ gives a surface. So there are infinitely many tangent lines. So we must specify a direction. The most obvious choices are the x-direction and y-direction.

Fix $y=b$, then the partial derivative with respect to x at (a,b) is

$$f_x(a,b) = g'(a), \quad \text{where } g(x) = f(x,b).$$

Similar notion for y.

Notations for partial derivatives. $z=f(x,y)$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_1 f = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = D_2 f = D_y f.$$

In order to find a partial derivative wrt x, we regard all other variables as constants, and take the derivative like in calc 1.

Example 3 Find f_x and f_y.

(a) $f(x,y) = y^3 - x^2y$

(b) $f(x,y) = x^2 e^{-y}$

(c) $f(x,y) = \tan^{-1}(xy^2)$

Solution

(a) $f_x = 0 - 2xy = -2xy$, $f_y = 3y^2 - x^2$

(c) $f_x = \dfrac{y^2}{1+x^2y^4}$, $f_y = \dfrac{2xy}{1+x^2y^4}$

(b) $f_x = 2xe^{-y}$, $f_y = -x^2e^{-y}$

Higher derivatives

We can consider the second partial derivatives. $z = f(x,y)$, then

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Example 4 Find the second partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2.$$

Solution $f_x = 3x^2 + 2xy^3$, $f_y = 3x^2y^2 - 4y$

$f_{xy} = 6xy^2$, $f_{xx} = 6x + 2y^3$, $f_{yy} = 6x^2y - 4$, $f_{yx} = 6xy^2$

Notice $f_{xy} = f_{yx}$. In fact,

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Can also talk about higher order derivatives like
f_{xxx}, f_{xyx}, f_{xxyx}, etc.

<u>Example 5</u> Let

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then it can be shown that for $(x,y) \neq (0,0)$, we have

$$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

and $f_x(0,0) = f_y(0,0) = 0$.

It can also be shown that $f_{xy}(0,0) = -1$, but $f_{yx}(0,0) = 1$.

This does not contradict Clairaut's Theorem, however, because for $(x,y) \neq (0,0)$,

$$f_{xy} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3},$$

which is <u>not</u> continuous at $(0,0)$.