Instructions. Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

Problem 1. (10 points) Find all vectors $\mathbf{v}$ such that

(a) $\langle 2,1,3\rangle \times \mathbf{v} = \langle 1,2,-3\rangle$

(b) $\langle 2,1,3\rangle \times \mathbf{v} = \langle 4,1,-3\rangle$

Solution. Let $\mathbf{v} = \langle a,b,c\rangle$. Then

$$\begin{aligned}\langle 2,1,3\rangle \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ a & b & c \end{vmatrix} \\ &= \langle c-3b, 3a-2c, 2b-a\rangle \,.\end{aligned}$$

(a) We are looking for solutions to the system of equations

$$\begin{aligned} c-3b &= 1 && (1)\\ 3a-2c &= 2 && (2)\\ 2b-a &= -3 && (3)\end{aligned}$$

Solving equations (1) and (3) for a and c, respectively, and substituting into (2), we find

$$\begin{aligned} 3a-2c &= 2\\ 3(3+2b)-2(1+3b) &= 2\\ 7 &= 2,\end{aligned}$$

which is a contradiction. So There are no solutions.

(b) Similarly, we are looking for solutions to the system

$$\begin{aligned} c-3b &= 4 && (4)\\ 3a-2c &= 1 && (5)\\ 2b-a &= -3 && (6)\end{aligned}$$

Now we solve (4) and (6) for a and c, respectively, and substitute this into (5). This gives

$$\begin{aligned} 3a-2c &= 1\\ 3(3+2b)-2(4+3b) &= 1\\ 1 &= 1\end{aligned}$$

Since $1=1$ is always true, our equation does not depend on b. Thus we can set $b=t$ arbitrarily, and the vectors of the form

$$\mathbf{v} = \langle 3+2t, t, 4+3t\rangle$$

for $t\in\mathbb{R}$ satisfy the given cross product. $\square$

Problem 2. (10 points)

(a) Find the equation for plane passes through $(3, -1, 4)$ and contain the line $\dfrac{x-1}{3} = \dfrac{y-3}{2}$, $z = 4$.

(b) Find the angle between the plane in last part and the plane $x + y + z = 2$.

Solution. (a) The direction numbers for the line are $3, 2, 0$, so the vector $\mathbf{a} = \langle 3, 2, 0\rangle$ is parallel to the plane. The point $(1, 3, 4)$ is on the line. This corresponds to $t = 0$ in the parametric equations for this line. The point $(3, -1, 4)$ is not on the line: if we plug this into the symmetric equations, we get $\frac{2}{3} = -\frac{4}{2} = -2$, which is not true.

Now we take the vector between these points: $\mathbf{b} = \langle 3-1, -1-3, 4-4\rangle = \langle 2, -4, 0\rangle$. Then $\mathbf{b}$ is also parallel to the plane, but $\mathbf{b}$ is not parallel to $\mathbf{a}$. So a normal vector for the plane is

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 3, 2, 0\rangle \times \langle 2, -4, 0\rangle = \langle 0, 0, -16\rangle$$

So an equation for the plane is

$$0(x-3) + 0(y+1) - 16(z-4) = 0,$$

which we can simplify to $z = 4$. Alternatively, we could have seen this by noting that the line is contained in the plane $z = 4$ and the point $(3, -1, 4)$ is in the plane $z = 4$.

(b) From (a) we have $\mathbf{n}_1 = \langle 0, 0, -16\rangle$. A normal vector to the plane $x+y+z = 2$ is $\mathbf{n}_2 = \langle 1, 1, 1\rangle$. Now

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)\\ &= \cos^{-1}\left(\frac{-16}{16\sqrt{3}}\right)\\ &= \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right).\end{aligned}$$

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