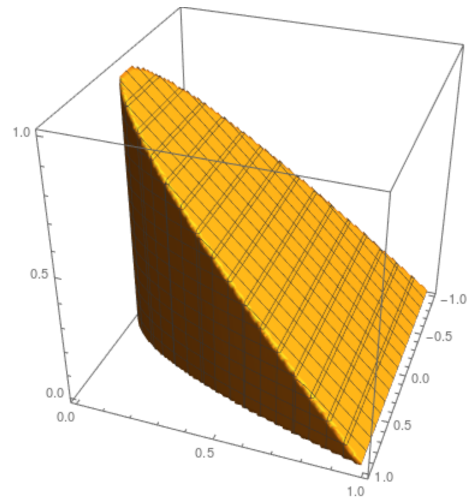
Quiz 6 — MA 261

Instructions. Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

Problem 1. Use a triple integral to find the volume of the solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$.

Solution. The region is plotted below.



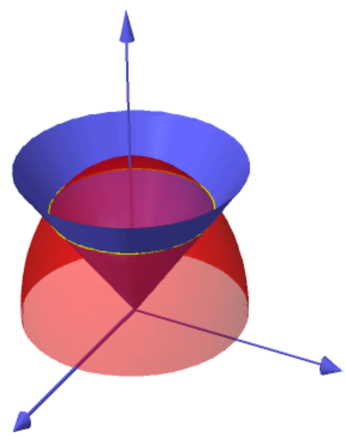
Here $E = \{(x, y, z) \mid 0 \le z \le 1 - y,\ x^2 \le y \le 1,\ -1 \le x \le 1\}$. So

$$\begin{aligned}
V(E) &= \iiint_E dA \\
&= \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} dz\, dy\, dx \\
&= \int_{-1}^{1} \int_{x^2}^{1} (1 - y)\, dy\, dx \\
&= \int_{-1}^{1} \Big[y - y^2/2 \Big]_{y=x^2}^{y=1} dx \\
&= \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{1}{2} x^4 \right) dx \\
&= 2 \int_{0}^{1} \left(\frac{1}{2} x - x^2 + \frac{1}{2} x^4 \right) dx \\
&= 2 \left[\frac{1}{2} x - \frac{1}{3} x^3 + \frac{1}{10} x^5 \right]_0^1 \\
&= \frac{8}{15}
\end{aligned}$$

□

Problem 2. Set up the integral $\iiint_E x dV$,where E is the solid above cone $z^2 = x^2 + y^2$ and below the sphere $x^2 + y^2 + z^2 = 1$ using cylindrical coordinates. (Do not compute the integral.)

Solution. It helps to have a picture in mind.



To use cylindrical coordinates, notice that $0 \le \theta \le 2\pi$. Now z ranges from the cone to the sphere, so $\sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}$. In cylindrical coordinates, this becomes $r \le z \le \sqrt{1-r^2}$. To find the bounds for r, we need to find the equation of that yellow circle of intersection. This occurs when $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = 1$. That is, $2(x^2 + y^2) = 1$. So $r^2 = 1/2$, and the bounds for r become $0 \le r \le 1/\sqrt{2}$. So the integral becomes

$$\iiint_E x\,dV = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \int_r^{\sqrt{1-r^2}} r^2 \cos\theta\, dz dr d\theta \qquad \square$$

Notice that we could also do this in spherical coordinates: It's easy to tell that $0 \le \theta \le 2\pi$ and $0 \le \rho \le 1$. Since we are above the cone and below the sphere, $0 \le \phi \le \pi/4$. (Recall that the cone $z = \sqrt{x^2+y^2}$ is $\phi = \pi/4$ in spherical coordinates.) So the integral becomes

$$\begin{aligned}\iiint_E x\,dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho \sin\phi \cos\theta \cdot \rho^2 \sin\phi\, d\rho d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^3 \sin\phi \cos\theta \sin\phi\, d\rho d\phi d\theta\end{aligned}$$