**Instructions.** Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

**Problem 1.** Use Greens Theorem to find the work done by the force

$$\mathbf{F} = \langle x^3 - y, x + \ln(y) \rangle$$

on a particle which moves once around the circle $x^2 + y^2 = 1$ in the positive direction.

*Solution.* Let $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$. Then by Green's Theorem,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \left( \frac{\partial}{\partial x}(x + \ln(y)) - \frac{\partial}{\partial y}(x^3 - y) \right) dA \\ &= \iint_D 2\, dA \\ &= 2A(D) = 2\pi.\end{aligned}$$

□

**Problem 2.** Verify that Green's Theorem is true for the line integral

$$\int_C xy^2\, dx - x^2 y\, dy$$

where $C$ consists of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-1, 1)$.

*Solution.* Let $D$ be the region enclosed by the parabola and the line segment. First using Green's Theorem:

$$\begin{aligned}\int_C xy^2\, dx - x^2 y\, dy &= \iint_D \left( \frac{\partial}{\partial x}(-x^2 y) - \frac{\partial}{\partial y}(xy^2) \right) dA \\ &= \int_{-1}^{1} \int_{x^2}^{1} -2xy\, dy\, dx \\ &= \int_{-1}^{1} \Big[ -xy^2 \Big]_{y=x^2}^{y=1} dx \\ &= \int_{-1}^{1} \Big( -1 + x^5 \Big) dx \\ &= 0,\end{aligned}$$

where in the last line we use the fact that $y = x^5 - 1$ is an odd function.

Calculating the integral directly, let $C_1$ be the part of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and let $C_2$ be the segment from $(1, 1)$ to $(-1, 1)$. For $C_1$

we have $y = x^2$, so $dy = 2x\,dx$, and

$$\begin{aligned}\int_{C_1} xy^2\,dx - x^2y\,dy &= \int_{-1}^{1} x(x^2)^2\,dx - x^2(x^2)(2x)\,dx \\ &= \int_{-1}^{1} (x^5 - 2x^5)\,dx \\ &= \int_{-1}^{1} -x^5\,dx \\ &= 0\end{aligned}$$

since $y = -x^5$ is an odd function. For the curve $C_2$, we have $y = 1$, so $dy = 0$. Thus

$$\int_{C_2} xy^2\,dx - x^2y\,dy = \int_{-1}^{1} x(1)^2\,dx = 0$$

So the integrals agree as we expect from Green's Theorem. $\square$