Instructions. Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

Problem 1. (10 points) Determine whether or not

$$\mathbf{F}(x,y,z) = 12x^2\,\mathbf{i} + \cos y\cos z\,\mathbf{j} + (1-\sin y\sin z)\,\mathbf{k}$$

is conservative vector field. If it is conservative, find a potential function for $\mathbf{F}$. (A potential function for $\mathbf{F}$ is a function f such that $\mathbf{F} = \nabla f$.)

Solution. Computing the curl of $\mathbf{F}$,

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &= \nabla\times\mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 12x^2 & \cos y\cos z & 1-\sin y\sin z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(1-\sin y\sin z) - \frac{\partial}{\partial z}(\cos y\cos z)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(1-\sin y\sin z) - \frac{\partial}{\partial z}(12x^2)\right)\mathbf{j} \\
&+ \left(\frac{\partial}{\partial x}(\cos y\cos z) - \frac{\partial}{\partial y}(12x^2)\right)\mathbf{k} \\
&= (-\cos y\sin z + \cos y\sin z)\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \\
&= \mathbf{0}
\end{aligned}$$

Since the domain of $\mathbf{F}$ is $\mathbb{R}^3$, we can conclude that $\mathbf{F}$ is conservative.

Now

$$f_x = 12x^2 \quad f_y = \cos y\cos z \quad f_z = 1-\sin y\sin z$$

So,

$$\begin{aligned}
f(x,y,z) &= 4x^3 + g(y,z) \\
&\implies f_y = g_y = \cos y\cos z \\
&\implies f(x,y,z) = 4x^3 + \sin y\cos z + h(z) \\
&\implies f_z = -\sin y\sin z + h'(z) = 1-\sin y\sin z \\
&\implies h'(z) = 1 \\
&\implies h(z) = z + K
\end{aligned}$$

Taking $K = 0$, we have $f(x,y,z) = 4x^3 + \sin y\cos z + z$. □

Problem 2. (10 points) Identify the surface with the vector equation

$$\mathbf{r}(u,v) = \langle 2+u, 3-u-v, v+5\rangle.$$

Solution. We have

$$x = 2+u \tag{1}$$
$$y = 3-u-v \tag{2}$$
$$z = v+5 \tag{3}$$

Solving for u and v in (1) and (3), then substituting into (2), we get

$$y = 3 - (x - 2) - (z - 5),$$

which is a plane. We could rewrite this equation as $x + y + z = 10$. $\square$