1. Complex numbers. Complex numbers are expression of the form $a+bi$, where $a, b \in \mathbf{R}$, which are added and multiplied as polynomials with respect to the letter i, with the convention that $i^2 = -1$. The set of all complex numbers is denoted by $\mathbf{C}$. Usually a complex number is denoted with one letter, for example $z = x + iy$, $\zeta = \xi + i\eta$ or $w = u + iv$.

1.1 Verify that $\mathbf{C}$ is a field. Hint: $(a + bi)^{-1} = a/(a^2 + b^2) - ib/(a^2 + b^2)$.
1.2 There is a one-to-one correspondence between complex numbers $a + ib$ and vectors $(a, b) \in \mathbf{R}^2$, which gives a geometric interpretation of addition of complex numbers.
1.3 Consider the set of matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{where} \quad a \text{ and } b \quad \text{are real.}$$

Show that the usual operations of addition and multiplication of matrices preserve this set of matrices, and that it forms a field isomorphic to $\mathbf{C}$. This gives a geometric interpretation of multiplication of complex numbers: describe it in words. (Hint: what is the geometric meaning of a matrix?)

We use shorthand notations like $a + i0 = a$, $a + i1 = a + i$ and so on. Then $\mathbf{R} \subset \mathbf{C}$, and the operations in $\mathbf{C}$ agree with the usual ones in $\mathbf{R}$. We describe such situation by saying that $\mathbf{R}$ is a subfield of $\mathbf{C}$, or that $\mathbf{C}$ is an extension of $\mathbf{R}$.

A one to one map σ of a field into itself is called an automorphism if it has the properties $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$.

1.4 The only automorphisms of $\mathbf{C}$, that satisfy $\sigma(x) = x$ for every $x \in \mathbf{R}$, are the identity and the automorphism $\sigma(a+ib) = a-ib$. The last automorphism is called *complex conjugacy* and the standard notation for it is $\bar{z}$. So if $z = x + iy$ then $\bar{z} = x - iy$.

If $z = x + iy$ is a complex number, then the real numbers $\Re z := x$ and $\Im z := y$ are called *real* and *imaginary* parts of z. We have

$$\Re z = \frac{1}{2}(z + \bar{z}) \quad \text{and} \quad \Im z = \frac{1}{2i}(z - \bar{z}). \tag{1}$$

The expression

$$z\bar{z} = (\Re z)^2 + (\Im z)^2 \tag{2}$$

is always non-negative, as a sum of squares of real numbers, so one can consider the non-negative square root of it: $|z| := \sqrt{z\bar{z}}$. This is called the *absolute value* or *norm* or *modulus* of z. From (2) follows

$$|\Re z| \le |z|, \quad |\Im z| \le |z| \quad \text{and} \quad |z| \le |\Re z| + |\Im z|. \tag{3}$$

1.5 Absolute value has the following properties:

a) $|z| \ge 0$ with equality iff $z = 0$,
b) $|z_1 z_2| = |z_1||z_2|$,
c) $|z_1 + z_2| \le |z_1| + |z_2|$.

Hint for c): Write $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = |z_1|^2 + |z_2|^2 + 2\Re(z_1 \bar{z}_2)$, using (1). Then use the first inequality in (3).

If complex numbers are interpreted as vectors in $\mathbf{R}^2$, then $|,|$ is the Euclidean norm in $\mathbf{R}^2$. Whenever we have a norm, a *distance* between two vectors can be defined as $\rho(z_1, z_2) = |z_1 - z_2|$.

1.6 The distance function has the following properties:

a) $\rho(z_1, z_2) \ge 0$ with equality iff $z_1 = z_2$ (positivity),
b) $\rho(z_1, z_2) = \rho(z_2, z_1)$ (symmetry),
c) $\rho(z_1, z_2) \le \rho(z_1, z_3) + \rho(z_2, z_3)$ (triangle inequality).

A set with a function satisfying a), b) and c) is called a metric space.

1.7 The subset $\{z : |z| = 1\} \subset \mathbf{C}$ is called *the unit circle*. Every complex number can be written as a product of a non-negative number and a number on the unit circle. The ratio of any two numbers on the unit circle, again belongs to the unit circle. This shows that the unit circle is a subgroup of the multiplicative group of $\mathbf{C}$.
1.8 The expression $\Re(z_1 \bar{z}_2)$ gives the dot-product in $\mathbf{R}^2$. The operation $(z_1, z_2) \mapsto z_1 \bar{z}_2$ is called the *Hermitian product*. Thus the dot product is the real part of the Hermitian product. Does the imaginary part of the Hermitian product look familiar to you?

Consider a plane with rectangular coordinates, and to every point (a, b) in this plane put into correspondence the complex number $a + ib$. This correspondence is one-to-one, which gives a third geometric interpretation to complex numbers. It also consistent with the notion of distance: the distance between two points in the plane is defined by the same formula as the distance between two complex numbers. Whenever one has a notion of a distance, satisfying a), b), and c) from 1.6, all topological notions can be defined (neighborhoods, open and closed sets, limits and continuity). For example, Let E be a set in the complex plane, and a is a point in E. We say that E is a *neighborhood* of a if $\{z : |z - a| < r\} \subset E$ for some $r > 0$.

1.9 Intersection of a finite set of neighborhoods of a is again a neighborhood of a. Union of *any* set of neighborhoods of a is again a neighborhood of a.

A set is called *open* if it is a neighborhood of each of its points. Telling which subsets are open defines a topology on a set. Topology of the set of complex numbers is the same as the topology in the plane $\mathbf{R}^2$, because the notion of the distance is the same. See any introductory text of topology, or an advanced calculus textbook. We only recall some facts and notions.

A *sequence* is a function on the set of non-negative (sometimes only positive) integers. It is usually denoted as (z_n), or

$$(z_n)_{n=0}^{\infty}, \quad \text{or simply} \quad z_0, z_2, z_3, \ldots .$$

We say that a sequence of complex numbers (z_n) has a limit $a \in \mathbf{C}$ if *for every neighborhood E of a there exists an integer N, such that $z_n \in E$ for every $n \geq N$.*

1.10 $z_n \to a$ iff $\Re z_n \to \Re a$ and $\Im z_n \to \Im a$. Hint: use inequalities (3).

A sequence (z_n) is called *Cauchy sequence* if *for every $\epsilon > 0$ there exists such integer N that $\rho(z_m, z_n) \leq \epsilon$ for all $m \geq N$ and $n \geq N$.* (This can be defined for any metric space). A metric space is called *complete* is every Cauchy sequence has a limit.

1.11 Prove that $\mathbf{C}$ is complete. Hint: Use 1.9 and the fact that $\mathbf{R}$ is complete (in the axiomatic definition of $\mathbf{R}$ this is actually one of the axioms!)
1.12 A topological space X is called *connected* if the only subsets which are simultaneously open and closed are $\emptyset$ and X. Prove that $\mathbf{C}$ is connected. Hint: consider $\mathbf{R}$ first.
1.13 If $f : X \to Y$ is a surjective continuous map of topological spaces, and X is connected then Y is also connected. As a corollary we obtain that the unit circle T is connected.

A set which is open and connected is called a *region.*