Math 511, Final exam, Fall 2024

NAME: ID number:

1. For the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

perform the Gram–Schmidt orthogonalization of the columns, and find an orthogonal matrix Q and an upper triangular matrix R, so that

$$A = QR.$$

What is the element $r_{2,3}$ of the matrix R?

A. 0
B. 1
C. 2
D. 3/2
E. 2/3

Ans. C. Hint: Q is a permutation matrix in this example.

2. Suppose that $n-1$ columns of an $n \times n$ real orthogonal matrix are given. How many possibilities are there for the remaining column?

A. One
B. Two
C. Infinitely many
D. It depends of the $n-1$ given columns
E. None of the above

Ans: B. Explanation: we have $n-1$ linear equations for the last column, saying that it is orthogonal to the rest. The null-space of this system has dimension 1, and we have the normalization condition that the norm of this last column is 1. This leaves exactly two possibilities for this last column: it can be multiplied by ± 1.

3. Evaluate the determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}.$$

A. 0.
B. 1.
C. 4.
D. -2.
E. None of the above.

Ans. A. You can do this either by row operations, or expanding on the first row (or 1-st column) which gives a simple recurrent relation.

4. Solve the difference equation

$$2a_{n+2} = 3a_{n+1} - a_n, \quad a_0 = 1, \quad a_1 = 2.$$

What is y_{2024}?

A. 1.
B. 2^{2024}.
C. $3 - 2^{-2023}$
D. $1 + 2^{-2024}$.
E. None of the above.

Ans. C. The general colution is $c_1 + c_2.2^{-n}$.

5. Solve the system of differential equations:

$$\mathbf{y}' = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{y}, \quad \mathbf{y}(0) = (1,1)^T.$$

What is $\mathbf{y}(1)$?

A. $(e + e^2, e^2)^T$.
B. $(e, e^2)^T$.
C. $(e + e^{-1}, e)^T$
D. $(e^2, e^2)^T$.
E. None of the above.
Ans.: D. The solution satisfying the initial condition is $y(t) = (1,1)^T e^{2t}$.

6. What is the signature of this quadratic form

$$x^2 + y^2 + z^2 - 2xy - 2yx - 2xz\ ?$$

A. $(+, +, +)$
B. $(+, +, -)$
C. $(+, -, -)$
D. $(+, -, 0)$
E. $(+, 0, 0)$

Ans.: B.

7. Find all values of a such that $\lambda = 1$ is an eigenvalue of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ a & 3 & 4 \\ 0 & a & 5 \end{pmatrix}.$$

A. $a = 0$.
B. $a = 1/3$.
C. $a = 0$ and $a = 8/3$.
D. $a = 1$ and $a = 2/3$.
E. None of the above.

Ans.: C

8. What are the lengths of major and minor semi-axes of the ellipse

$$2x^2 + 2xy + 2y^2 = 1 \quad ?$$

A. 3 and 1.
B. 1 and $1/\sqrt{3}$.
C. 1 and $1/3$.
D. $\sqrt{3}$ and 1.
E. None of the above.

Ans.: B

True or false:

9. There exists a 2×2 matrix A such that $A \neq 0$ but $A^2 = 0$.

Ans.: T. For example, the Jordan cell with eigenvalue 0.

10. There exists a 2×2 matrix A such that $A^2 \neq 0$ but $A^3 = 0$.

Ans.: F. Proof: Bring it to the Jordan form. If $A^3 = 0$, all eigenvalues are 0. Now consider all 3 possibilities (one Jordan cell, two or three). In all cases $A^2 = 0$.

11. If A and B are real symmetric matrices then AB is symmetric.

Ans.: F. $(AB)^T = B^T A^T = BA$ which is in general not the same as AB.

12. If A and B ares real orthogonal matrices then AB is orthogonal.

Ans.: T.

13. If A and B are real symmetric positive definite matrices then $A + B$ is positive definite.

Ans.: T.

14. For a real symmetric matrix, eigenvalues are the same as singular values.

Ans.: F. Singular values are always no-negative, while eigenvalues of a symetric matrix can be arbitrary real numbers.

15. For every (rectangular) matrix A, A and $A^T A$ have the same null-space.

Ans.: T. This was proved in class 2 times.