MATH 511, Final exam, Spring 2012

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1. Circle the letters corresponding to statements which are true for every pair of $n \times n$ real matrices A and B

A. If A and B are symmetric then $2A - B$ is symmetric.

B. If A and B are symmetric then AB is symmetric.

C. If A and B are orthogonal then $A - B$ is orthogonal.

D. If A and B are orthogonal then BA is orthogonal.

E. If A and B are positive definite then $2A + B$ is positive definite.

Ans. A,D,E.

2. Circle the letters corresponding to the statements which are true for all 5×5 real diagonalizable matrices A:

A. A has 5 distinct eigenvalues.
B. A has at least one real eigenvalue.
C. A is non-singular.
D. There exists an orthogonal basis in $\mathbf{R}^5$ consisting of eigenvectors.
E. $P(A)$ is diagonalizable for every polynomial P

Ans. B,E.

3. Circle letters corresponding to the statements which are true for all 5×5 matrices A and B:

A. $\det(7A) = 7\det(A)$.
B. $\det(A + B) = \det(A) + \det(B)$.
C. $\mathrm{tr}(A - B) = \mathrm{tr}(A) - \mathrm{tr}(B)$.
D. $\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)$.
E. $\exp(A + B) = \exp(A)\exp(B)$.

Ans. C.

4. True or false:

a) If a basis consists of eigenvectors of a real symmetric matrix then this basis must be orthogonal.

b) If A is real, symmetric and orthogonal, then $A^2 = -I$.

c) If the only eigenvalue of a matrix A is 0 then A is the zero-matrix.

d) If the only eigenvalue of a real symmetric matrix is 1 then $A = I$.

e) Every non-singular matrix is diagonalizable.

Ans. Only d) is true, the rest are false.

5. Consider the differentiation operator D on the space V_d of all polynomials of degree at most d.

a) Find the eigenvalues and eigenvectors.

b) Is this operator diagonalizable? If not, find the Jordan form.

Solution. The eigenvalue problem is equivalent to solving

$$p' = \lambda p, \quad p \neq 0,$$

where p is a polynomial of degree at most d. This is a differential equation whose general solution is

$$p(t) = Ce^{\lambda t}.$$

This can be a polynomial only when $\lambda = 0$. In which case, $p(t) = C$, a constant. So the only eigenvalue of our operator is $\lambda = 0$ and the eigenspace has dimension 1 (consists of constant polynomials).

Therefore the Jordan form consists of a single cell of size $d+1$ (dimension of the space), it is denoted by $J_{d+1}(0)$. This matrix has zeros on the main diagonal, ones on the adjacent diagonal above the main, the rest are zeros.

Of course, an alternative way to solve this problem is to write the matrix of differentiation in some basis, find the eigenvalues and eigenvectors.

6. Find the signature (the number of positive and negative squares in the canonical form) of the quadratic form $x^T Ax$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 1 & 7 \end{pmatrix}.$$

Ans. $(+, +, -)$. For example by doing row operation and finding the signs of pivots.

7. Evaluate the determinant with any method:

$$\begin{vmatrix} 7 & 7 & 7 & 7 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{vmatrix}.$$

Ans. -21.

8. Solve the differential equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}, \quad \text{with} \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The answer should not contain complex numbers.

Solution. The characteristic equation is

$$\lambda^2 - 2\lambda + 2 = 0,$$

so $\lambda = 1 \pm i$.

For $\lambda = 1 + i$ the eigenvector is $(1, i)^T$. (The eigenvector to the other, conjugate eigenvalue is the conjugate $(1, -i)^T$, but we don't need it. From the first eigenvalue and its eigenvector we obtain a complex solution

$$e^{(1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + ie^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

The real and imaginary parts of it are two linearly independent real solutions, so the general real solution can be written as

$$x(t) = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Plugging $t = 0$ and using the initial condition, we conclude that $c_1 = 3$ and $c_2 = 1$. Therefore the answer is

$$x(t) = e^t \begin{pmatrix} 3\cos t + \sin t \\ -3\sin t + \cos t \end{pmatrix}.$$

9. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

Ans.

$$A^{-1} = \begin{pmatrix} 1/2 & -1/2 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & -3/2 & 1/2 \end{pmatrix}.$$