On an entire function

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November 27, 2005

The functional equation

$$F'(z) = F(qz), \quad F(0) = 1,$$

has a unique series solution in powers of z:

$$F(z) = \sum_{n=0}^{\infty} q^{n(n-1)/2} z^n / n!.$$

When $|q| \leq 1$, this series represents an entire function. We are interested in the location of zeros of F for $|q| < 1$. The case $|q| = 1$ was recently investigated in [1] where the references related to this case are also given. We denote these zeros by z_j, so that

$$|z_0| \leq |z_1| \leq \ldots. \tag{1}$$

Here each zero is repeated according to its multiplicity. When $q \in [0, 1]$, all zeros are negative and simple. This follows from a result of Laguerre Pólya and Schur on multiplier sequences [2, 4]. Langley [3] proved for $q \in (0, 1)$ that the sequence z_{n+1}/z_n is decreasing for n large enough, and furthermore,

$$z_n/z_{n-1} = (n + 1)/(nq) + o(n^{-2}), \quad n \to \infty,$$

from which follows

$$z_n = -nq^{-n}(\gamma + o(1)), \quad n \to \infty, \tag{2}$$

where γ is a constant.

*Supported by the NSF.

In this note we prove, that (2) holds for all complex q in the open unit disc, with $\gamma = 1$. In particular, all zeros but finitely many are simple. As an application of the method, we will also show that inequalities (1) are strict when $|q| < 0.39$. In particular, all zeros are simple, for such q.

It will be convenient to work with a modified function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} q^{n^2} z^n / n!, \quad |q| < 1, \tag{3}$$

which is related to F by the formula

$$f_q(z) = F_{q^2}(qz). \tag{4}$$

Let $|q| < 1$ be fixed, and let N be a large integer. (How large will depend on $|q|$.) The N-th term of the series (3) is maximal (among all the terms of this series) when

$$r_{N-1} < |z| < r_N,$$

where

$$r_N = (N+1)|q|^{-2N-1}.$$

We will restrict $|z|$ to this annulus, or maybe to a slightly larger annulus, so we put

$$z = wN/q^{2N},$$

so that w is near 1. Setting $n = N + m$, we have

$$\frac{f(z)}{c_N z^N} = \sum_{m=-N}^{\infty} \frac{N^m N!}{(N+m)!} w^m q^{m^2}. \tag{5}$$

We will compare this with the theta-function

$$\theta_3(w, q) = \sum_{m=-\infty}^{\infty} w^m q^{m^2}.$$

The idea is that:
a) only few terms in both sums are relevant, and
b) for theta functions, we know both their coefficients and their zeros percisely.

First of all, the factor

$$\frac{N^m N!}{(N+m)!} < 1 \quad \text{for all} \quad m \neq 0, \tag{6}$$

both positive and negative. So it can be neglected in the estimates of the tails of the series.

Second, we will take a partial sum of (5) for $|m| < \mu$ where $\mu = \mu(q)$ is independent of N. So our estimates will be valid when $N > \mu$. Notice that for $|m| < \mu$, the factor (6) tends to 1 as $N \to \infty$, so it can be neglected.

To choose μ, we want the tail of the theta-function to be less than minimum of its modulus for some fixed $|w|$. We have the Euler–Jacobi product (following the notation of Whittaker-Watson):

$$\begin{aligned} \theta_3 &= \prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}\left(1+q^{2n-1}(w+w^{-1})+q^{4n-2}\right) \\ &= \prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}(1+q^{2n-1}w)(1+q^{2n-1}/w). \end{aligned}$$

So the zeros are at $q^{2n+1} : -\infty < n < \infty$. The minimum on the unit circle is at least

$$c(q) := \prod_{n=1}^{\infty}(1-|q|^{2n})\prod_{n=1}^{\infty}(1-|q|^{2n-1})^2,$$

which is $\theta_3(-1, |q|)$, that is

$$c(q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n|q|^{n^2} > 0. \tag{7}$$

Let us also introduce $c_1(q)$, the minimum of the θ_3 on the circle $|w| = |q|^2$. This minimum is also positive,

$$c_1(q) = |q|^{-1}c(q) > c(q). \tag{8}$$

Now we will define $\mu = \mu(q)$ by the condition

$$4\sum_{m\geq\mu}|q|^{m^2} < \epsilon\min\{c(q), c_1(q)\} = \epsilon c(q),$$

where $\epsilon \in (0,1)$. Then equation (5) will give us

$$\frac{f(z)}{c_N z^N} = \theta_3(w) + \text{error term}, \tag{9}$$

where the error term is less than the minimum of θ_3 on the boundary of the ring

$$A = \{w : |q|^2 < |w| < 1\}.$$

Furthermore, the error term tends to zero as $N \to \infty$, and the rate of this convergence can be also estimated. As θ_3 has exactly one zero in the annulus A, namely at the point $-q$, Rouche's theorem implies that f has one zero in the annulus

$$N|q|^{-2N-2} < |z| < N|q|^{-2N},$$

and this zero is within $\epsilon N|q|^{-2N}$ from the point

$$Z_N = -Nq^{-2N+1}.$$

Let us also show that f has exactly N zeros in the disc

$$|z| < N|q|^{-2N}. \tag{10}$$

The number of zeros of F in this disc is equal to the increment I of the arg f on the circle $|z| = N|q|^{-2N}$, divided by 2π. According to (5), this increment I is $2\pi N$ plus the increment of the argument of θ_3 on the unit circle. But the increment of the argument of θ_3 on the unit circle is zero (this is because it is zero for very small q, and depends continuously on q). Thus $I = 2\pi N$ and f has exactly N zeros in the disc (10).

So the N-th zero of f is close to Z_N. Translated in terms of the zeros of the original function F, whis gives

$$z_n = -(n + o(n))q^{-n},$$

as advertized.

In particular, this gives an upper estimate for the N-th zero, and shows that no zero can escape to infinity while q varies on a compact subset of the punctured unit disc.

Now we separate the roots of F. To separate the root z_0 of the smallest modulus, we write

$$F(z) = (1 + z + qz^2/2) + (q^3z^3/6 + \ldots) = P(z) + Q(z).$$

The two roots of P are $(-1 \pm \sqrt{1-2q})/q$; if $|q| < 1/2$, then they are separated by the circle $C = \{z : |z| = |q|^{-1}\}$ and we have $|P(z)| \geq 1/|2q| - 1$ on this circle. Now we estimate $|Q|$ from above on C:

$$|Q(e^{i\phi}/|q|)| \leq \sum_{n=3}^{\infty} |q|^{n(n-3)/2}/n! \leq \frac{1}{6(1-|q|^2)}.$$

Thus if

$$\frac{1}{|2q|} - 1 > \frac{1}{6(1-|q|^2)}, \tag{11}$$

then F has exactly one zero inside the circle C, and this is the case when

$$|q| < 0.41.$$

To prove separation of other roots, we choose $\mu = 2$ in the previous arguments, that is we write as in (5) with $N \geq 2$

$$\frac{f(z)}{c_N z^N} = 1 + q(\frac{N}{N+1}w + w^{-1}) + Q(w) = P(w) + Q(w).$$

The equation $P(z) = 0$ has two roots. If $|q| \leq 1/2$, these two roots are separated by the circle $|w| = \sqrt{(N+1)/N}$, and the minimum of $|P|$ on this circle is at least $1 - |q|\sqrt{N/(N+1)}$.

The maximum of $|Q(w)|$ on the same circle is at most $2|q|^4/(1-|q|)$. So the moduli of all zeros z_n for $n \geq 1$ will be separated if

$$(1 - |q|\sqrt{N/(N+1)})(1-|q|) > 2|q|^4,$$

that is $|q| \leq 0.436$.

References

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